Commun. Fac. Sci. Univ. Ank. Series A_1
V. 44. pp. 15-24 (1995)

THE SEIFERT - VAN KAMPEN THEOREM FOR THE GROUP OF GLOBAL SECTIONS

SABAHATTİN BALCI

Department of Math. Faculty of Sci. University of Ankara. 06100 *Tandoğan–Ankara–TURKEY*

(Received Aug 22, 1994; Accepted March 21, 1995)

SUMMARY

Let X be the union of the subspaces U_1 and U_2 that are both open, path connected, $U_{12} = U_1 \cap U_2 \neq \varnothing$ and U_{12} is also path connected. In this paper, We first contruct the sheaf H of the fundamental groups of a path connected space and give the characteristic features of H. Then, the homomorphisms and global sections of the sheaf H are explored. Finally it is proved that if the groups of global sections $\Gamma(U_{12}, H_{12}) = <S; R>, \Gamma(U_1, H_1) = <S_1; R_1>$ and $\Gamma(U_2, H_2) = <S_2; R_2>$ are given, then the group $\Gamma(X, H)$ is isomorphic to the group defined by the generators $S_2 \cup S_2$ and the relations $R_2 \cup R_2 \cup R_S$. As a result of this, the sheaf H, especially the fundamental group $\pi_1(X, x)$ was easily calculated for any $x \in X$.

1. INTRODUCTION

Let X be a path connected space and H_x be any fundamental group of X based for any $x \in X$, that is $H_x = \pi_1(X, x)$. Let X = (X, c) be a pointed topological space, for an arbitrary fixed point $c \in X$. Also, let $H = \bigvee_{x \in X} H_x$. H is a set over X and the mapping $\varphi: H \to X$ defined by $\varphi(\sigma_x) = \varphi([\alpha]_x) = x$, for any $\sigma_x = [\alpha]_x \in H_x \subset H$, is onto.

We introduce a topology on H as follows:

Let $W \subset X$ be an open set. Define a mapping $s: W \to H$ such that $s(x) = [(\gamma^{-1}\alpha)\gamma]_x$ for each $x \in W$, where $[\alpha]_c = \sigma_c \in H_c$ is any element and $[\gamma]$ is an arbitrary fixed homotopy class defines an isomorphism between H_x and H_c. Then, the change of s depends on only the change of $\sigma_c = [\alpha]_c$. Furthermore, $\varphi o s = 1_W$. Let us denote the totality of the mappings s defined on W by $\Gamma(W, H)$.

If B is a base for X, then $B^* = \{s(W): W \in B, s \in \Gamma(W, H)\}$ is a base for H. The mappings φ and s are continous in this topology.

Moreover, φ is a locally topological mapping. Then (H, φ) is a sheaf over X. (H, φ) (or only H) is called "The Sheaf of the Fundamental Groups" over X [1]. For any open set $W \subset X$, an element s of $\Gamma(W, H)$ is called a section of the sheaf H over W. The set $\Gamma(W, H)$ is a group with the pointwise operation of multiplication. Thus, H is a sheaf of groups over X [2]. Furthermore, the group $H_x = \pi_1(X, x)$ is called the stalk of the sheaf H for any $x \in X$.

2. CHARACTERISTIC FEATURES OF H [2].

2.1. Let $W \subset X$ be an open set. Then, any section over W can be extended to a global section over X. Furthermore,
$\Gamma(W, H) = \{s \mid W : s \in \Gamma(X, H)\} = \Gamma(X, H) \mid W$.

2.2. Any two stalks of H are isomorphic with each other.

2.3. Let $W_1, W_2 \subset X$ be any two open sets, $s_1 \in \Gamma(W_1, H)$ and $s_2 \in \Gamma(W_2, H)$. If $s_1(x_0) = s_2(x_0)$ for any point $x_0 \in W_1 \cap W_2$ then $s_1 = s_2$ over the whole $W_1 \cap W_2$.

2.4. Let $W \subset X$ be an open set and $s_1, s_2 \in \Gamma(W, H)$.

If $s_1(x_0) = s_2(x_0)$ for any point $x_0 \in W$, then $s_1 = s_2$ over the whole W.

2.5. To each point $\sigma_x \in H_x \subset H$, there uniquely corresponds a sec-sections $s \in \Gamma(W, H)$ such that $s(x) = \sigma_x$. Hence, $H_x \cong \Gamma(W, H)$. In partoicular, $H_x \cong \Gamma(X, H)$.

2.6. Let $x \in X$ be any point and $W = W(x)$ be any open set. Then, $\varphi^{-1}(W) = \bigvee_{i \in I} s_i(W), s_i \in \Gamma(W, H)$ and $\varphi \mid s_i(W) : s_i(W) \to W$ is a topological mapping for every $i \in I$. Hence, $W = W(x)$ is evenly covered by φ. Thus, φ is a cover projection and (H, φ) is a covering space of X. Moreover (H, φ) is regular, because the group T of cover transformations of H is isomorphic to the group H_x, that is T is transitive on H_x [2].

3. HOMOMORPHISMS AND THE GROUP $\Gamma(X, H)$

Let X_1, X_2 be topological spaces and H_1, H_2 be the corresponding sheaves, respectively. We begin by giving the following definition.

Definition 3.1. Let $f^*: H_1 \to H_2$ be a mapping. If f^* is continuous, a homomorphism on each stalk of H_1 and maps every stalk of H_1 into a stalk of H_2, then it is called a sheaf homomorphism.

Let f: $X_1 \to X_2$ be a continuous mapping and f*: $H_1 \to H_2$ be a sheaf homomorphism. If f* $(H_1)_{x_1} \subset (H_2)_{f(x_1)}$ for each $x_1 \in X_1$, then f* is (called) a stalk preserving mapping with respect to f.

Definition 3.2. Let f*: $H_1 \to H_2$ be a sheaf homomorphism. If f* is also a bijection, then f* is called a sheaf isomorphism.

Theorem 3.1. Let f: $X_1 \to X_2$ be a continuous mapping. Then there is a stalk preserving sheaf homomorphism f*: $H_1 \to H_2$ with respect to f.

Proof: Let $x_1 \in X_1$ be any point and α be a closed path based at x_1. Then f o α is a closed path with base point $f(x_1) = x_2$ in X_2 and $[f o \alpha] \in (H_2)_{x_2}$. On the other hand, if α_1 and α_2 are closed paths at x_1 in X_1 such that $\alpha_1 \backsim \alpha_2$, then $fo\alpha_1 \backsim fo\alpha_2$. Thus, we can define the mapping f*: $H_1 \to H_2$ such that $f^*(\sigma_{x_1}) = f^*([\alpha]_{x_1}) = [fo\alpha]_{f(x_1)=x_2}$ for any $[\alpha]_{x_1} = \sigma_x \in (H_1)_{x_1} \subset H_1$.

It is easily seen that the mapping f* is well-defined, stalk preserving with respect to f and homomorphism on each stalk [1].

To complete the proof, let us show that f* is continuous. Let $U_2 \subset f^*(H_1) \subset H_2$ be an open set. Without lost of generality, we assume that $U_2 = s^2(W_2)$, where $W_2 \subset X_2$ is an open set and $s^2 \in \Gamma(W_2, H_2)$. Thus, $\varphi_2(U_2) = \varphi_2(s^2(W_2)) = W_2$. Since f is continuous, $f^{-1}(W_2) = W_1 \subset X_1$ is an open set. Now let $\sigma_{x_2} \in U_2$ be an element. Then, there exists at least one element $\sigma_{x_1} \in U_1 = f^{*-1}(U_2)$ such that $f^*(\sigma_{x_1}) = \sigma_{x_2}$. Since $\varphi_1(\sigma_{x_1}) = x_1 \in W_1$, there is a section $s^1 \in \Gamma(W_1, H_1)$ such that $s^1(x_1) = \sigma_{x_1}$ and $s^1(W_1) \subset H_1$ is an open. Also $s^1(W_1) \subset U_1$. It is easily seen that $U = \bigcup_{i \in I} s_i^1(W_1)$. Therefore, $U_1 \subset H_1$ is an open set, that is f* is a continuous mapping.

Now, let $\mathcal{L}$ denote the category of connected topological spaces and continuous mappings and $\mathcal{D}$ denote the category of sheaves and sheaf homomorphisms. Let us define a mapping F: $\mathcal{L} \to \mathcal{D}$ such that $F(f) = f^*: H_1 \to H_2$ for any continuous mapping (morphism) f: $X_1 \to X_2$. Then,

1. If $f = 1_x$, then $F(1_x) = 1_{F(x)}$, since $(1_x)^* = [1_x o\alpha] = [\alpha]$ for any $\sigma_x = [\alpha]_x \in H_x$.
2. If $f_1: X_1 \to X_2$ and $f_2: X_2 \to X_3$ are any two morphisms, then $f_2 \text{ o } f_1 = f_2f_1: X_1 \to X_3$ is also a morphism and $F(f_2f_1) = (f_2f_1)^*: H_1 \to H_3$. However, $(f_2f_1)^*([\alpha]) = [(f_2f_1)\alpha]$, for any $[\alpha] \in H_{x_1} \subset H_1$. Since $(f_2f_1)\alpha \backsim f_2(f_1\alpha)$ rel. (0, 1), it can be written that $[(f_2f_1)\alpha] = [f_2(f_1\alpha)] = f_2^*([f_1\alpha]) = f_2^*(f_1^*([\alpha])) = (f_2^*f_1^*)([\alpha])$.

Theorem 3.2. There is a covaryant functor from the category of path connected topological spaces and continuous mappings to the category of sheaves and sheaf homomorphisms.

Let f: $X_1 \to X_2$ be now a topological mapping, then there exists the continous mapping $f^{-1}: X_2 \to X_1$ such that $ff^{-1} = 1_{X_2}$, $f^{-1} f = 1_{X_1}$.

From the theorem 3.1, there are the mappings $(f^{-1})^*: H_2 \to H_1$, $(ff^{-1})^* = (1_{X_2})^*: H_2 \to H_2$, $(f^{-1}f)^* = (1_{X_1})^*: H_1 \to H_1$

From the theorem 3.2, $(ff^{-1})^* = f^* (f^{-1})^* = 1_{F(X_2)}$, $(f^{-1}f)^* = (f^{-1})^* f^* = 1_{F(X_1)}$. Hence, $(f^{-1})^* = (f^*)^{-1}$. Thus, f^* is a sheaf isomorphism.

Corollary 3.1. Let f: $X_1 \to X_2$ be a topological mapping. Then the corresponding sheaves H_1 and H_2 are isomorphic.

Let f: $(X_1, c_1) \to (X_2, f(c_1) = c_2)$ be a continuous mapping. We know that the mapping $f^*: H_1 \to H_2$ is a sheaf homomorphism. Also, each element $\sigma_{c1} = [\alpha]_{c1} \in (H_1)_{c1}$ defines a unique section s^1 over X_1 such that $s^1(x_1) = [(\gamma^{-1} \alpha)\gamma]_{x_1}$, for any $x_1 \in X_1$. However, $f^*([\alpha]_{c1}) = [fo\alpha]_{c2} \in (H_2)_{c2}$ and $[fo\alpha]_{c2}$ defines a section s^2 over X_2 such that $s^2(x_2) = [(\delta^{-1} fo\alpha)\delta]_{x_2}$ for any $x_2 \in X_2$. Then the correspondence $[\alpha]_{c1} \leftrightarrow [fo\alpha]_{c2}$ bet ween $(H_1)_{c1}$ and $(H_2)_{c2}$ gives the correspondence $s_1 \leftrightarrow s_2$ between $\Gamma(X_1, H_1)$ and $\Gamma(X_2, H_2)$. If we denote this correspondence $f_*(s^1) = f^*([(\gamma^{-1}\alpha)\gamma]_{x_1}) = [(\delta^{-1} fo\alpha)\delta]_{x_2} = s^2(x_2)$, then the mapping $f_*: \Gamma(X_1, H_1) \to \Gamma(X_2, H_2)$ is a homomorphism. In fact, for any two sections $s^1_1, s^1_2 \in \Gamma(X_1, H_1)$ and any point $x_2 \in X_2$,

$$f_*(s^1_1)(x_2) = f^*([(\gamma^{-1}\alpha_1)\gamma]_{x_1}) = [(\delta^{-1} fo\alpha_1)\delta]_{x_2},$$

$$f_*(s^1_2)(x_2) = f^*([(\gamma^{-1}\alpha_2)\gamma]_{x_1}) = [(\delta^{-1} fo\alpha_2)\delta]_{x_2}, \text{ and}$$

$$(f_*(s^1_1).f_*(s^1_2)(x_2) = [(\delta^{-1} fo\alpha_1 . fo\alpha_2)\delta]_{x_2} = [(\delta^{-1} fo\alpha_1 . \alpha_2)\delta]_{x_2} = f_*(s^1_1 . s^1_2)(x_2).$$

We then state the following theorem.

Theorem 3.3. Let f: $X_1 \to X_1$ be a continuous mapping. Then there exists a homomorphism $f_*: \Gamma(X_1, H_1) \to \Gamma(X_2, H_2)$.

We now give the functorial statement of this theorem. Let $\mathcal{L}$ be the category of path connected topological spaces and continuous mapping and $\mathcal{D}$ be the category of groups and homomorphims. Let us define a mapping F: $\mathcal{L} \to \mathcal{D}$ with $F(X) = \Gamma(X, H)$ and $F(f) = f_*$ for any element $X \in \mathcal{L}$ and morphism $f: X_1 \to X_2$. F is a covaryant functor. In fact,

1. If $f = 1_x$, then $F(1_x) = (1_x)_*$ and $(1_x)_*(s) = s$ for any $s \in \Gamma(X, H)$.

Thus, $F(1_x) = 1_{F(x)}$.

2. Let $f_1: X_1 \to X_2$, $f_2: X_2 \to X_3$ morphisms. Then, $f_2 f_1 = f_2 \circ f_1: X_1 \to X_3$ is a morphism and $F(f_2 f_1) = (f_2 f_1)_*: \Gamma(X_1, H_1) \to \Gamma(X_3, H_3)$. Moreover, $(f_2 f_1)_*(s^1) = f_{2*}(f_{1*}(s^1)) = (f_{2*} f_{1*})(s^1)$. Hence, $F(f_2 f_1) = F(f_2)F(f_1)$.

We then state the following theorem.

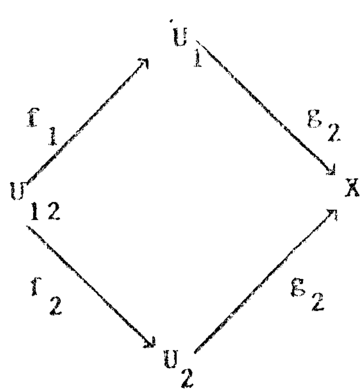
Theorem 3.4. There is a covaryant functor from the category of path connected topological spaces and continous mappings to the category of groups and homomorhisms.

Now, let $f_1: X_1 \to X_2$ be a topological mapping. Then there is the mapping $f^{-1}: X_2 \to X_1$ such that $ff^{-1} = 1_{X_2}$, $f^{-1}f = 1_{X_1}$. From theorems 3.3,3.4 $(ff^{-1})_* = f_*(f^{-1})_* = 1_{F(X_2)}$, $(f^{-1}f)_* = (f^{-1})_* f_* = 1_{F(X_1)}$. Hence, $(f^{-1})_* = (f_*)^{-1}$. Therefore f_* is an isomorphism. Notice that, for any $s^1 \in \Gamma(X_1, H_1)$ the composition $f^* \circ s^1 \circ f^{-1} \in \Gamma(X_2, H_2)$.

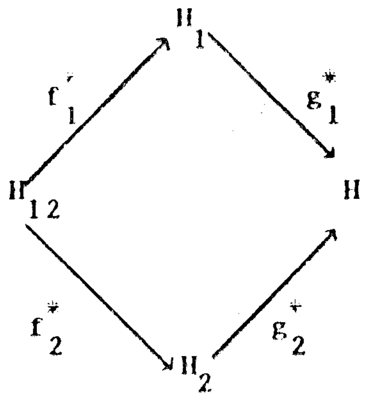
Corollary 3.2. Let $f: X_1 \to X_2$ be a topological mapping. Then, the corresponding groups $\Gamma(X_1, H_1)$ and $\Gamma(X_2, H_2)$ are isomorphic.

4. THE SEIFERT–VAN KAMPEN THEOREM FOR GLOBAL SECTIONS [3, 4, 5, 8]

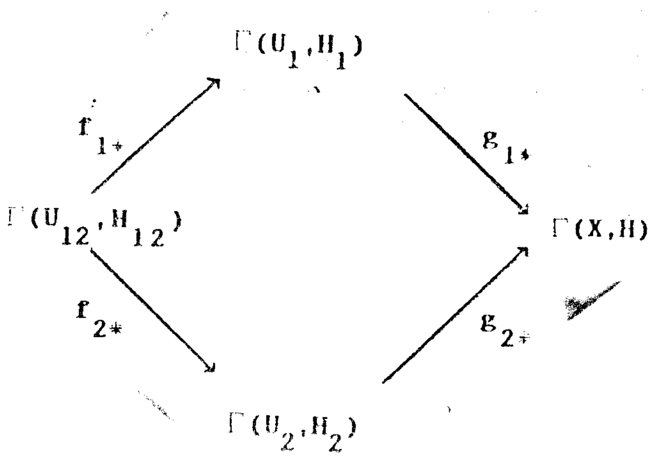
Let X be the union of the subspaces U_1 and U_2 which are both open, path connected and the intersection $U_{12} = U_1 \cap U_2 \neq \varnothing$ and U_{12} is also path connected. Let f_1, f_2, g_1, g_2 denote various inclusion mappings as indicated below



Form Theorem 3.1. We obtain the following diagram of homomorphisms defined on the corresponding sheaves of fundamental groups.



Recall that; H_{12}, H_1, H_2 and H are the sheaves which are constructed over U_{12}, U_1, U_2 and X, respectively. Hence, we can form the following diagram of homomorhisms defined on the groups of global sections.



Let us suppose that, $U_{12} = (U_{12}, c)$, $U_1 = (U_1, c)$, $U_2 = (U_2, c)$ for an arbitrary fixed point $c \in U_{12}$. Assume that the groups $\Gamma(U_{12}, H_{12}) = <S, R>$, $\Gamma(U_1, H_1) = <S_1 ; R_1>$ and $\Gamma(U_2, H_2)$

$= <S_2; R_2>$ are known. We will calculate the groups $\Gamma(X, H)$ by means of these groups.

Let R_s denote the following set of words $S_1 U S_2$:

$(f_{1*}s)(f_{1*}s)^{-1}$, $s \in S$.

We shall think of R_s as a set of relators. As a set of relations $R_s = \{f_{1*} s = f_{2*} s: s \in S\}$. We assert that the group $\Gamma(X, H)$ is isomorphic to the group defined by the generators $S_1 \cup S_2$ and the relations $R_1 \cup R_2 \cup R_s$. Note that the relations R of $\Gamma(U_{12}, H_{12})$ are not required. Loosely speaking $\Gamma(X, H)$ is the smallest group generated by $\Gamma(U_1, H_1)$ and $\Gamma(U_2, H_2)$ for which f_{1*} s $= f_{2*}$ s, $s \in \Gamma(U_{12}, H_{12})$.

To prove this assertion we begin by giving the following lemma.

Lemma 4.1. Let $\alpha: I \to X$ be a path and $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n = 1$. If the mapping $\alpha_i: I \to X$ defined by $\alpha_i(t) = \alpha((1-t)t_{i-1} + tt_i)$ for $i = 1, 2, \dots, n$ then $[\alpha] = [\alpha_1][\alpha_2] \dots [\alpha_n]$.

Proof: The proof is by induction on n. Suppose first that $n = 2$, then $0 = t_0 \leq t_1 \leq t_2 = 1$ and

$$(\alpha_1 . \alpha_2)(t) = \begin{cases} \alpha_1(2t), & o \leq t \leq 1/2 \\ \alpha_2(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

$$= \begin{cases} \alpha(2tt_1), & 0 \leq t \leq 1/2 \\ \alpha((1-(2t-1)t_1 + 2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

We can see that $\alpha_1 . \alpha_2 \backsim \alpha$ simply by using the homotopy

$F: I \times J \to X$ given by

$$F(t, s) = \begin{cases} \alpha((1-s)2tt_1 + st), & o \leq t \leq 1/2 \\ \alpha((1-s)(t_1 + (2t_1-1)(1-t_1)) + st), & 1/2 \leq t \leq 1 \end{cases}$$

Suppose now that $n > 2$ and the result holds for smaller integer. We have, $0 = t_0 \leq t_1 \leq \dots \leq t_n = 1$. Since $0 = t_0 \leq t_{n-1} \leq t_n = 1$ we can apply the above result to get $\alpha \backsim \beta\, \alpha_n$. where $\beta(t) = \alpha(tt_{n-1})$.

Now, $0 = \dfrac{t_0}{t_{n-1}} \leq \dfrac{t_1}{t_{n-1}} \leq \dots \leq \dfrac{t_{n-1}}{t_{n-1}} = 1$, so that by the inductive hypothesis, $[\beta] = [\beta_1][\beta_2] \dots [\beta_{n-1}]$, where

$\beta_i(t) = \beta\,((1-t)t_{i-1}/t_{n-1} + tt_1/t_{n-1})$

$= \alpha\,((1-t)t_{i-1} + tt_i) = \alpha_i(t)$.

Thus $[\alpha] = [\alpha_1][\alpha_2] \dots [\alpha_n]$, which competes the proof.

Let us now choose the paths q_i: $I \to X$ so that $q_i(0) = c$, $q_i(1) = \alpha(t_i)$ and so that $q_i(t) \in U_{12}$ for all $t \in I$ and for $i = 1, 2, \ldots, n-1$. Also, let q_0 and q_n be given by $q_0(t) = q_n(t) = c$.

Since $[\alpha] = [\alpha_1] [\alpha_2] \ldots [\alpha_n]$, we have

$$[\alpha] = [q_0] [\alpha_1] [q_1^{-1}] [q_1] [\alpha_2] [q_2^{-1}] \ldots [q_{n-1}] [\alpha_n] [q_n^{-1}]$$
$$= [(q_0\alpha_1) q_1^{-1}] [(q_1\alpha_2) q_2^{-1}] \ldots [(q_{n-1}\alpha_n) q_n^{-1}]$$

and each of $q_i (\alpha_{i+1}) q^{-1}{}_{i+1}$ are closed paths based c which lie entirely in U_1 or U_2. Hence $[(q_i\alpha_{i+1})q^{-1}{}_{i+1}]$ defines a section either in $\Gamma(U_1, H_1)$ or in $\Gamma(U_2, H_2)$ for $i = 1, 2, \ldots n-1$, so that for $\lambda(k) = 1$ or 2 and for $x_{\lambda(k)} \in U_{\lambda(k)}$,

$$s^{\lambda(k)} (x_{\lambda(k)}) = [(\gamma^{-1} (q_i\alpha_{i+1}) q^{-1}{}_{i+1}) \gamma] x_{\lambda(k)}$$

For brevity, let $(q_i\alpha_{i+1}) \; q^{-1}{}_{i+1} = \delta_{i+1}$. Thus, we can write that $[\alpha] = [\delta_1] [\delta_2] \ldots [\delta_n]$ such that each $[\delta_i]$ defines a section either in $\Gamma(U_1, H_1)$ or in $\Gamma(U_2, H_2)$. Also, the homotopy class $[\alpha]$ defines a section s in $\Gamma(X, H)$, that is $s(x) = [\gamma^{-1} \alpha) \gamma]_x$ for each $x \in X$. $[\alpha] = [\delta_1] [\delta_2] \ldots [\delta_n]$ implies that $s(x) = [(\gamma^{-1} \delta_1\delta_2 \ldots \delta_n) \gamma]_x$ for each $x \in X$. On the other hand, for any $x \in X$ and for $i = 1, 2, \ldots n$,

$s^i(x) = [(\gamma^{-1} \delta_i)\gamma]_x$ and it is defined that

$(s^i.s^k) (x) = s^i(x). s^k(x)$, thus

$$(s^1. s^2 \ldots s^n) (x) = s^1(x). s^2(x) \ldots s^n(x)$$
$$= [(\gamma^{-1} \delta_1)\gamma]_x . [(\gamma^{-1} \delta_2)\gamma]_x \ldots [(\gamma^{-1} \delta_n)\gamma]_x$$
$$= [(\gamma^{-1} \delta_1 . \delta_2 \ldots \delta_n)\gamma]_x = s(x).$$

Hence, each element of $\Gamma(X, H)$ may be written as the product of images of elements from $\Gamma(U_1, H_1)$ or $\Gamma(U_2, H_2)$ under g_{1*} or g_{2*}, respectively.

Corollary 4.1. The group $\Gamma(X, H)$ is generated by the set $g_{1*} (S_1) \cup g_{2*} (S_2)$ where S_1, S_2 are the generators of $\Gamma(U_1, H_1)$, $\Gamma(U_2, H_2)$, respectively.

From the definition of g_{i*} we can identify S_i with $g_{i*}(S_i)$ for $i = 1, 2$. In this sense $\Gamma(X, H)$ is generated by $S_1 \cup S_2$ where S_1, S_2 generate $\Gamma(U_1, H_1)$, $\Gamma(U_2, H_2)$ respectively.

Lemma 4.2. The generators of $\Gamma(X, H)$ statisfy the relations R_1, R_2 and R_s. Moreover R_1, R_2 and R_s are the unique relations in $\Gamma(X, H)$.

Proof: Since g_{i*}: $\Gamma(U_i, H_i) \to \Gamma(X, H)$ is homomorphism for $i = 1, 2$ any relation satisfied by the elements of S_i in $\Gamma(U_i, H_i)$ is also satisfied by the elements g_{i*} $(S_i) \subset \Gamma(X, H)$. Thus ,if we use our convention of suppressing g_{i*}, the elements $S_1 \cup S_2$ in $\Gamma(X, H)$ satisfy the relations R_1 and R_2.

If $s \in S \subset \Gamma(U_{12}, H_{12})$ then $g_{1*}\ f_{1*}\ s = g_{2*}\ f_{2*}\ s$ since $g_1 f_1 = g_2 f_2$. If a word in S_i represents f_{i*} s, then the same word in S_i represents $g_{i*}\ f_{i*}$ s in $\Gamma(X, H)$ so that $f_{1*}\ s = f_2\ s$, $s \in S$, and so the proof of the first part of lemma 4.2. is finished.

Let us now suppose that $s = s_1^{\in(1)}\ s_2^{\in(2)} \ldots s_k^{\in(k)} = I$ is a relation between the elements of $S_1 \cup S_2 \subset \Gamma(X, H)$, Here $\in(i) = \mp 1$ and $s_i \in S_{\lambda(1)}$ for $i = 1, 2 \ldots, k$ where $\lambda(i) = 1$ or 2. From the definition of the elements of $\Gamma(X, H)$ there is a unique element $[\alpha]$ and unique homotopy classes $[\alpha_i]$ such that $[\alpha]$ defines the sections s and each of $[\alpha_i]$ define the sections s_i. Thus, for $i=1, 2, \ldots, k$

$$[\alpha] = [\alpha_1]^{\in(1)} \ . \ [\alpha_2]^{\in(2)} \quad \ldots \ [\alpha_k]^{\in(k)} = [1].$$

However, it has been proved in [4, 6, 7] that $[\alpha]$ can be reduce to $[1]$ by a finite sequence of operations each of which inserts or delates an expression from a certain list. Hence s is a consequence of the relations $R_1 \cup R_2 \cup R_s$ and $R_1 \cup R_2 \cup R_s$ are the unique relations in $\Gamma(X, H)$.

As a result of lemma 4.1. and 4.2. we can state that

Corollary 4.2. The group $\Gamma(X, H)$ is isomorphic to the group defined by the generators $S_1 \cup S_2$ and the relations $R_1 \cup R_2 \cup R_s$.

W then state the following theorem.

Theorem 4.1. (The Seifert–Van Kampen Theorem For Global Sections). Let us suppose that the topological space X is the union of the subspaces U_1 and U_2 which are both open, path connected, $U_{12} = U_1 \cap U_2 \neq \varnothing$ and U_{12} is also path connected. Let the groups $\Gamma(U_{12}, H_{12})$, $\Gamma(U_1, H_1)$ and $\Gamma(U_2, H_2)$ be known. Then, i) (The "generators" of $\Gamma(X, H)$). If $s \in \Gamma(X, H)$ is any section, then

$$s = \prod_{k=1}^{n} g_{\lambda(k)*}\ s_k, \text{ where } s_k \in \Gamma(U_{\lambda(k)}, H_{\lambda(k)}),\ \lambda(k) = 1 \text{ or } 2$$

ii) (The "relators" or "relations" of $\Gamma(X, H)$).

Let $s = \prod_{k=1}^{n} g_{\lambda(k)*}\ s_k \in \Gamma(X, H)$. Then $s = I$ if and only if s can be reduced to I by a finite sequence of operations each of which inserts or delates an expression from a certain list.

It we restrict this theorem to any stalk $H_x \subset H$ for any $x \in X$, we get the known Seifert–Van Kampen Theorem at once such that it does not depend on the base point.

Theorem 4.2. (The Seifert–Van Kampen Theorem for the fundamental groups). Let us suppose that the topological space X satisfy the conditions mentioned in theorem 4.1 and let the groups $\Gamma(U_{12}, H_{12})$, $\Gamma(U_1, H_1)$ and $\Gamma(U_2, H_2)$ be known. Then,

i) (The "generators" of $\pi_1(X, x)$.) If $x \in X$ is any point and $[\alpha] \in \pi_1(X, x)$ is any element, then $[\alpha] = \prod_{k=1}^{n} g_{\lambda(k)*} s_k(x)$, where $s_k \in \Gamma(U_{\lambda(k)}, H_{\lambda(k)})$, $\lambda(k) = 1$ or 2.

ii) (The "relators" or "relations" of $\pi_1(X, x)$.)

Let $[\alpha] = \prod_{k=1}^{n} g_{\lambda(k)*} s_k(x)$. Then $[\alpha] = [1]$ if and only if $[\alpha]$ can be reduced to $[1]$ by a finite sequence of operations each of which inserts or delates an expression from a certain list.

REFERENCES

[1] S. BALCI., "On the sheaf of the homology groups of the Complex Manofold". Jour. Inst. Math.-Comp. Sci. Math. Ser. Vol. 1. No. 2. pp. 127–133, 1988 (INDIA).

[2] S. BALCI., "On the Restricted Sheaf". Comm. Fac. Sci. Univ. Ankara Ser, Al, Vol. 37, pp. 41–51, 1988.

[3] R.H. CROWELL., "On the Van Kampen Theorem". Pac. J. Math., pp. 43–50, 1959.

[4] C. GODBILLON., "Elements de Topologie Algebraique". Paris: Herman 1971.

[5] R.J. KNILL., "The Seifert–Van Kampen Theorem Via Regular Covering Speces". Pasific. J. Math., 49, pp. 149–160, 1973.

[6] W.S. MASSEY., "Algebraic Topology an Introduction" Springer–Verlag, Newyork, 1977.

[7] H. SEIFERT., "Konstruktion Dreidimensionaler Geschlossener Raume" Ber. Sachs. Akad. Wiss. 83, pp. 26–66, 1931.

[8] E.R. VAN KAMPEN., "On the connection between the fundamental groups of some related spaces". Amer. J. Math., 55, pp. 261–267, 1933.