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QUARTER-SYMMETRIC METRIC CONNECTION IN AN SP-SASAKIAN MANIFOLD

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ABSTRACT

The object of the present paper is to study some properties of curvature tensor of a quarter-symmetric metric connection in an SP-Sasakian manifold.

1. INTRODUCTION

Let M^n be an n-dimensional C^∞ - manifold. If there exists a tensor field F of type (1,1), a vector field ξ and a 1-form η in M^n satisfying

$$\bar{\bar{X}} = X - \eta(X)\xi, \quad \bar{X} = F(X), \quad \eta(\xi) = 1, \tag{1.1}$$

then M^n is called an almost paracontact manifold.

Let g be the Riemannian metric satisfying

$$\eta(X) = g(X,\xi) \tag{1.2}$$

$$\eta(FX) = 0,\ f\xi = 0,\ \text{rank}\ (F) = n-1 \tag{1.3}$$

$$g(FX,FY) = g(X,Y) - \eta(X)\,\eta(Y) \tag{1.4}$$

The set (F, ξ, η, g) satisfying (1.1), (1.2), (1.3) and (1.4) is called an almost paracontact Riemannian structure. The manifold with such a structure is called an almost paracontact Riemannian manifold [3].

If we define $'F(X,Y) = g(\bar{X},Y)$, then in addition to the above relations the followings are satisfied;

$$'F(X,Y) = {}'F(Y,X) \tag{1.5}$$

$$'F(\bar{X},\bar{Y}) = {}'F(X,Y) \tag{1.6}$$

Let us consider an n-dimensional differentiable manifold M with a positive definite metric g which admits a 1-form η satisfying

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0 \tag{1.7}$$

and

$$(\nabla_X \nabla_Y \eta)(Z) = - g(X,Z)\, \eta(Y) - g(X,Y)\, \eta(Z) + 2\eta(X)\, \eta(Y)\, \eta(Z) \tag{1.8}$$

where ∇ denotes the covariant differentiation with respect to g. Moreover, if we put

$$\eta(X) = g(X,\xi),\ \nabla_X \xi = \bar{X}, \tag{1.9}$$

then it can be easily verified that the manifold in consideration becomes an almost pracontact Riemannian manifold. Such a manifold is called a P-Sasakian manifold [1].

For a P-Sasakian manifold M, the following relations hold;

$$\eta(R(X,Y)Z) = g(X,Z)\, \eta(Y) - g(Y,Z)\, \eta(X), \tag{1.10}$$

$$S(X,\xi) = - (n-1)\, \eta(X), \tag{1.11}$$

where R and S are the curvature tensor and the Ricci tensor respectively.

Now, we consider an n-dimensional differentiable manifold M with a Riemannian metric g which admits a 1-form η satisfying

$$(\nabla_X \eta)(Y) = - g(X,Y) + \eta(X)\, \eta(Y). \tag{1.12}$$

By putting $\eta(X) = g(X,\xi)$ and $(\nabla_X \eta)(Y) = 'F(X,Y)$, we can easily show that the manifold in consideration is a P-Sasakian manifold. Such a manifold is called an SP-Sasakian manifold [1]. Thus for such a SP-Sasakian manifold, we have

$$'F(X,Y) = - g(X,Y) + \eta(X)\, \eta(Y) \tag{1.13}$$

A linear connection $\widetilde{\nabla}$ in a Riemannian manifold M^n is said to be a quarter-symmetric connection if its torsion tensor T satisfies

$$T(X,Y) = \eta(Y)\ \varphi(X) - \eta(X)\ \varphi(Y) \tag{1.14}$$

where η is a 1-form and φ is a (1,1) tensor field [2]. A linear connection $\tilde{\nabla}$ is called a metric connection, iff

$$\left(\tilde{\nabla}_X g\right)(YZ) = 0 \tag{1.15}$$

A linear connection $\tilde{\nabla}$ satisfying (1.14) and (1.15) is called a quarter-symmetric metric connection [5].

If $\varphi(X) = X$, then the connection is called a semi-symmetric metric connection [5]. The semi-symmetric metric connection in an SP-Sasakian maniold have been studied by Sinha and Kalpana [4]. In the present paper we have studied with the quarter-symmetric metric connection in an SP-Sasakian manifold. In section 2 we have deduced the expressions for the curvature tensor and the Ricci tensor of M^n with respect to the quarter-symmetric metric connection. Some properties of the curvature tensor with respect to the quarter-symmetric metric connection have been studied. In general, the Ricci tensor of the quarter-symmetric metric connection is not symmetric. Here it is proved that in an SP-Sasakian manifold the Ricci tensor of the quarter-symmetric metric connection is symmetric. Also, in general, the conformal curvature tensors of the quarter-symmetric metric connection and the Riemannian connection are not equal. Finally it is proved that in an SP-Sasakian manifold the conformal curvature tensors of the quarter-symmetric metric connection and the Riemannian connection are equal and also it is proved that if the curvature tensor of the quarter-symmetric metric connection vanishes then the manifold is conformally flat.

2. CURVATURE TENSOR

We consider $\varphi(X)$ as a contact structure $F(X) = \bar{X}$ in equation (1.14). The manifold M^n is considered to be an SP-Sasakian manifold. The equation (1.14) and (1.15) can be written as

$$T(X,Y) = \eta(Y)\bar{X} - \eta(X)\bar{Y} \tag{2.1}$$

$$\left(\tilde{\nabla}_X g\right)(YZ) = 0 \tag{2.2}$$

Let $\tilde{\nabla}$ be a linear connection and ∇ be a Riemannian connection such that

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X,Y) \tag{2.3}$$

where U is a tensor of type (1.2). For $\tilde{\nabla}$ to be a quarter-symmetric metric connection in M^n, we have from

$$U(X,Y) = \frac{1}{2}\left[T(X,Y) + T'(X,Y) + T'(Y,X)\right], \tag{2.4}$$

where

$$g(T'(X,Y),Z) = g(T(Z,X),Y) \tag{2.5}$$

(See [5]).

From (2.1) and (2.5) we get

$$T'(X,Y) = \eta(X)\bar{Y} - {}'F(X,Y)\xi \tag{2.6}$$

where $'F(X,Y) = g(\bar{X},Y)$, η is a 1-form and ξ is the associated vector field.

From (2.1), (2.4) and (2.6) we have

$$U(X,Y) = \eta(Y)\bar{X} - {}'F(X,Y)\xi \tag{2.7}$$

From (2.3) and (2.7) we get

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\bar{X} - {}'F(X,Y)\xi \tag{2.8}$$

Hence a quarter-symmetric metric connection $\tilde{\nabla}$ in an SP-Sasakian manifold is given by (2.8).

Let $\tilde{R}$ and R be the curvature tensors of the connections $\tilde{\nabla}$ and ∇ respectively. Then we have

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z \tag{2.9}$$

and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2.10}$$

Using (2.8) and (2.10) in (2.9) we have

$$\tilde{R}(X,Y)Z = R(X,Y)Z + 3'F(X,Z)\overline{Y} - 3'F(Y,Z)\overline{X}$$
$$+ \left[(\nabla_X F)(Y) - (\nabla_Y F(X))\right]\eta(Z) -$$
$$+ \left[(\nabla_X{}'F)(Y,Z) - (\nabla_Y{}'F(X,Z))\right]\xi \qquad (2.11)$$

Equation (2.11) can be written as

$$\tilde{R}(X,Y,Z,U) = {}'R(X,Y,Z,U) + 3'F(X,Z)'F(Y,U) - 3'F(Y,Z)'F(X,U)$$
$$+ \left[(\nabla_X F)(Y,U) - (\nabla_Y{}'F(X,U))\right]\eta(Z) -$$
$$- \left[(\nabla_X{}'F)(Y,Z) - (\nabla_Y{}'F(X,Z))\right]\eta(U) \qquad (2.12)$$

where $'\tilde{R}(X,Y,Z,U) = g(\tilde{R}(X,Y)Z,U)$ and $'R(X,Z,Y,U) = g(R(X,Y)Z,U)$

Since

$$(\nabla_X{}'F)(Y,Z) = \nabla_X{}'F(Y,Z) - {}'F(\nabla_X Y,Z) - {}'F(Y,\nabla_X Z) \qquad (2.13)$$

then comparing with (2.13) we get

$$(\nabla_X{}'F)(Y,Z) = {}'F(X,Y)\,\eta(Z) + {}'F(X,Z)\,\eta(Y)$$

So

$$(\nabla_X{}'F)(Y,Z) - (\nabla_X{}'F)(X,Z) = {}'F(X,Z)\,\eta(Y) \quad {}'F(Y,Z)\,\eta(X) \qquad (2.14)$$

From (1.13), (2.14) and (2.12) we get

$$\tilde{R}(X,Y,Z,U) = {}'R(X,Y)Z,U) + 3g(X,Z)g(Y,U) - 3g((Y,Z)g(X,U)$$
$$- 2g(X,Y)\,\eta(Y)\,\eta(U) - 2g(Y,U)\,\eta(X)\,\eta(Z)$$
$$+ 2g(Y,Z)\,\eta(X)\,\eta(U) + 2g(X,U)\,\eta(Y)\,\eta(Z) \qquad (2.15)$$

A relation between the curvature tensor of M^n with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and the Riemannian connection ∇ is given by the equation (2.15). Putting $X = U = e_i$ in (2.15) where $\{e_i\}$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \le i \le n$ we get

$$\tilde{S}(Y,Z) = S(Y,Z) - (3n-5)\,g(Y,Z) + 2(n-2)\,\eta(Y)\,\eta(Z) \qquad (2.16)$$

where $\tilde{S}$ and S are the Ricci tensors of the connection $\tilde{\nabla}$ and ∇

respectively. Again, putting $Y = Z = e_i$ in (2.16) we get

$$\tilde{r} = r - (n-1)\,(3n-4) \tag{2.17}$$

where $\tilde{r}$ and r are the scalar curvatures of the connections $\tilde{\nabla}$ and ∇ respectively.

Theorem 1. For a SP-Sasakian manifold M with quarter-symmetric metric connection $\tilde{\nabla}$, we have

(a) $\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0$
(b) $'\tilde{R}(X,Y,Z,U) + {}'\tilde{R}(X,Y,U,Z) = 0$
(c) $'\tilde{R}(X,Y,Z,U) - {}'\tilde{R}(Z,U,X,Y) = 0$
(d) $'\tilde{R}(X,Y,Z,\xi) = 2\,'R(XY,Z,\xi)$
(e) $\tilde{S}(X,\xi) = 2S(X,\xi)$

Proof. Using (2.15) and the first Binachi identity with respect to the Riemannian connection, we have (a). From (2.15) we get (b) and (c). Putting $U = \xi$ in (2.15) and using (1.10) we get (d). Putting $Y = Z = e_i$ in (d) and taking summation over i, we get (e).

Theorem 2. In an SP-Sasakian manifold M the Ricci tensor of the quarter-symmetric metric connection is symmetric.

Proof. The proof of the theorem obviously follows from (2.16).

Weyl conformal curvature tensor C of type (0,4) of M^n with respect to the Riemannian connection is given by

$$\begin{aligned} C(X,Y,Z,U) = {}'R(X,Y,Z,U) - \frac{1}{n-2}\,[&S(Y,Z)\,g(X,U) - S(X,Z)\,g(Y,U) \\ &+ S(X,U)\,g(Y,Z) - S(Y,U)\,g(X,Z)] + \\ &+ \frac{r}{(n-1)\,(n-2)}\,[g(Y,Z))\,g(X,U) - g(X,Z)\,g(Y,U)] \end{aligned} \tag{2.18}$$

Analogous to this definition, we define conformal curvature tensor of M^n with respect to the quarter-symmetric metric connection by

$$\tilde{C}(X,Y,Z,U) = {}'\tilde{R}(X,Y,Z,U) - \frac{1}{n-2}\,[\tilde{S}(Y,Z)\,g(X,U) - \tilde{S}(X,Z)\,g(Y,U)$$

$$+ \tilde{S}(X,U)\, g(Y,Z) - \tilde{S}(Y,U)\, g(X,Z)\big] +$$
$$+ \frac{\tilde{r}}{(n-1)\,(n-2)} \,[g(Y,Z))\, g(X,U) - G(X,Z)\, g(Y,U)] \qquad (2.19)$$

From (2.15), (2.16), (2.17), (2.18) and (2.19), we have

$$\tilde{C}(X,Y,Z,U) = C(X,Y,Z,U) \qquad (2.20)$$

Hence we can state the following theorem.

Theorem 3. In an SP-Sasakian manifold the conformal curvature tensors of the quarter-symmetric metric connection and the Riemannian connection are equal.

Let us now consider $\tilde{R} = 0$. Then we have $\tilde{S} = 0$ and $\tilde{r} = 0$ and hence from (2.19) we get $\tilde{C} = 0$. So, from (2.20) we get $C = 0$.

Thus we have the following theorem:

Theorem 4. If in an SP-Sasakian manifold the curvature tensor of a quarter-symmetric metric connection vanishes, then the manifold is conformally flat.

If $\tilde{S} = 0$, then from (2.16) we get

$$S(Y,Z) = (3n-5)\, g(Y,Z) - 2(n-2)\, \eta(Y)\, \eta(Z) \qquad (2.21)$$

Since $\tilde{S} = 0$, $\tilde{r} = 0$, and so from (2.17) we get

$$r = (n-1)\,(3n-4) \qquad (2.22)$$

From (2.15), (2.18), (2.21) and (2.22) we get

$$'\tilde{R}(X,Y,Z,U) = C(X,Y,Z,U)$$

Hence we can state the following theorem:

Theorem 5. If in an SP-Sasakian manifold the Ricci tensor of a quarter-symmetric metric connection $\tilde{\nabla}$ vanishes, then the curvature tensor of $\tilde{\nabla}$ is equal to the conformal curvature tensor of the manifold.

From Theorem 4 and Theorem 5 we have the following theorem:

Theorem 6. If in an SP-Sasakian manifold the Ricci tensor of a quarter-symmetric metric connection $\widetilde{\nabla}$ vanishes, then the manifold is conformally flat iff the curvature tensor with respect to the quarter-symmetric metric connection vanishes.

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