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HIGHER ORDER GAUSSIAN CURVATURES OF PARALLEL HYPERSURFACES

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ABSTRACT

Let M be a hypersurface in (n+1) -dimensional Euclidean space E^{n+1} and $\overline{M}$ be a parallel hypersurface to M.

The higher order Gaussian curvatures of $\overline{M}$ are known ([1], [2]).

In this paper we give the higher order Gaussian curvatures of $\overline{M}$ by using its principal curvatures and a new lemma 2.1.

Ali GÖRGÜLÜ's paper [5] gives the same results. But he uses a different method. So, our study offers a new and direct method.

1. INTRODUCTION

Normal curvatures, principal curvatures for hypersurfaces and the relavent higher order Gaussian curvatures are invariants independent of the choice of coordinates.

There has been some recent studies on the relations between higher order Gaussian curvatures of a hypersurface M in E^{n+1} and that of another hypersurface $\overline{M}$ which is parallel to M [5].

These invariants, relavent definitions and theorems are generalized by replacing E^{n+1} with a (n+1) -Riemanniann manifold [7].

Using the definition of Gaussian curvatures in [5] and [7], geometric interpretations and results along with some relations are obtained.

In this work, although the same definition is used, an original lemma is stated and proved by induction. Thus, this paper introduces a brand

new lemma and a new method to give the relations in between the higher order Gaussian curvatures for the parallel hypersurfaces.

In each step of the induction deals with a unique hypersurface. For example, when n = 5, for the calculations of relations between 3rd degree Gaussian curvature, we considered a 5-dimensional hypersurface in E^6.

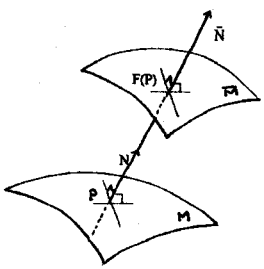
The new method and the lemma introduced in this work is not only applicable to the parallel hypersurface pairs but also is applicable to the pedal hypersurface pairs and invers hypersurface pairs, as well.

Evidently, κ_i^p and κ_i^{p+1}, of the lemma, are distinct dimensional i-th curvatures and there is no doubt about the additions of them, geometricaly or algebraicaly.

Let M and $\overline{M}$ be two hypersurfaces in E^{n+1}. If N is the unit normal vector field of M then $\overline{N}$ is the unit normal vector field of $\overline{M}$ such that $\overline{N}$ is parallel translated vector field of N on $\overline{M}$, then we have that

$$N = \sum_{i=1}^{n+1} a_i \frac{\partial}{\partial x_i},$$

where each a_i is a c^∞ function on M.



If there is a function

$f: M \to \overline{M}$

$P \to F(P) = \overrightarrow{OP} + rN.$

So, the coefficients of $\overline{N}$ satisfy that

$\bar{a}_i(F(P)) = a_i(P), \forall P \in M,$

Then $\overline{M}$ is called a parallel hypersurface to M, where $r \in IR$ is a constant [3].

Theorem 1.1. Let M and $\overline{M}$ are two hypersurfaces, such that $\overline{M}$ is parallel hypersurface to M. Then for $X \in \chi(M)$ and $\overline{X} \in \chi(\overline{M})$, we have

1. $F_*(X) = \overline{X} + r\overline{S(X)}$.

2. $\overline{S}(F_*X) = \overline{S(X)}$.

3. If k is a principal curvature of M at the point P in the direction X, then $\frac{k}{1 + rk}$ is the principal curvature of $\overline{M}$ at the point f(P) in direction $F_*(X)$, that is,

$$\bar{S}(F_*X) = \frac{k}{1+rk} F_*(X)$$

which means that F preserves principal directinos, where F_* is the diferential of F [4].

2. CALCULATIONS OF $\bar{\kappa}_j^{(n)}$ IN TERMS OF $\kappa_j^{(n)}$

Let M and $\bar{M}$ are two hypersurfaces in E^{n+1}, such that $\bar{M}$ be parallel to M, where k_j, $1 \le j \le n$ are the principal curvatures in directions, X_j, $1 \le j \le n$ at the point P of M and $F_*(X_j)$ are the principal directions of $\bar{M}$ at the point F(P) then we know [6] that:

$$\bar{S}(F_*X_1) = \frac{k_1}{1+rk_1} F_*X_1,$$

$$\bar{S}(F_*X_2) = \frac{k_2}{1+rk_2} F_*X_2,$$

$$\vdots$$

$$\bar{S}(F_*X_n) = \frac{k_n}{1+rk_n} F_*X_n.$$

Thus, we have

$$\bar{S}(F_*X_j) = \frac{k_j}{1+rk_j} F_*X_j, \quad 1 \le j \le n,$$

where $\bar{S}$ is the shape operator of $\bar{M}$ and so

$$\bar{k}_j = \frac{k_j}{1+rk_j} \tag{1}$$

are the principal cunvatures of $\bar{M}$, [4].

According to equation (1) and we know the higher order Gaussian curvatures of M ([1], [2]) by

$$\kappa_1^{(n)} = \sum_{j=1}^{n} k_j,$$

$$\kappa_2^{(n)} = \sum_{j_1<j_2=1}^{n} k_{j_1} k_{j_2},$$

$$\kappa_p^{(n)} = \sum_{j_1<j_2<\dots<j_p=1}^{n} k_{j_1} k_{j_2} \dots k_{j_p},$$

$$\vdots$$

$$\kappa_n^{(n)} = \prod_{j=1}^{n} k_j,$$

where k_{j_r} denotes j_r - th principal curvature functions on M, $1 \le j_r \le n$.

The curvatures $\kappa_1^{(n)}$ and $\kappa_n^{(n)}$ are the Gaussian and mean curvature functions on M.

In a similar way, we verify directly that the higher order Gaussian curvatures of $\overline{M}$ are defined by:

$$\overline{\kappa}_1^{(n)} = \sum_{j=1}^{n} \overline{k}_j,$$

$$\overline{\kappa}_2^{(n)} = \sum_{j_1<j_2=1}^{n} \overline{k}_{j_1}\overline{k}_{j_2},$$

$$\overline{\kappa}_p^{(n)} = \sum_{j_1<j_2<\dots<j_p=1}^{n} \overline{k}_{j_1}\overline{k}_{j_2}\dots\overline{k}_{j_p},$$

$$\vdots$$

$$\overline{\kappa}_n^{(n)} = \prod_{j=1}^{n} \overline{k}_j,$$

where $\overline{k}_{j_r}$, denotes the j_r-th principal curvature functions on $\overline{M}$ and $\overline{k}_j = \dfrac{k_j}{1+rk_j}$ $1 \le j \le n$, [4].

As a result of these relations, we give the theorem 2.2 for the higher order Gaussian curvatures of parallel hypersurfaces, which is proved by induction method and using the following lemma 2.1.

Lemma 2.1. Let M be a hypersurface of E^{n+1}. There are the following relations between the higher order Gaussian curvature functions and the principal curvature functions of M:

(a) $\kappa_1^{(p)} + k_{p+1} = \kappa_1^{(p+1)}$, $1 \le p+1 \le n$,

(b) $\kappa_p^{(p)} + k_{p+1} = \kappa_{p+1}^{(p+1)}$, $1 \le p+1 \le n$,

(c) $\kappa_r^{(p)} + k_{p+1} \cdot \kappa_{r-1}^{(p)} = \kappa_r^{(p+1)}$, $1 \le p+1 \le n$,

Proof:

(a) According to the definition, we have

$$\kappa_1^{(p)} = k_1 + k_2 + \dots + k_p \quad , 1 \le p \le n,$$

$$\kappa_1^{(p+1)} = k_1 + k_2 + \dots + k_p + k_{p+1},$$
$$= \kappa_1^{(p)} + k_{p+1}.$$

(b) From the definition

$$\kappa_p^{(p)} = \sum_{j_1<j_2<\dots<j_p=1}^{n} k_{j_1} k_{j_2} \dots k_{j_p}, \quad 1 \le p \le n.$$

We multiply both sides of this equation by k_{p+1} then we have

$$\kappa_p^{(p)} k_{p+1} = \left(\sum_{j_1<j_2<\dots<j_p}^{p} k_{j_1} k_{j_2} \dots k_{j_p} \right) k_{p+1},$$
$$= \sum_{j_1<j_2<\dots<j_p}^{p} k_{j_1} k_{j_2} \dots k_{j_p} k_{p+1},$$
$$= \sum_{j_1<j_2<\dots<j_p<j_{p-1}}^{p+1} k_{j_1} k_{j_2} \dots k_{j_p} k_{j_{p-1}},$$
$$= \kappa_{p+1}^{(p+1)}.$$

(c) From the definition we have

$$\kappa_r^{(p+1)} = \sum_{j_1<j_2<\dots<j_r=1}^{p+1} k_{j_1} k_{j_2} \dots k_{j_r}.$$

or

$$\kappa_r^{(p+1)} = \left(\sum_{j_1<j_2<\dots<j_r=1}^{p} k_{j_1} k_{j_2} \dots k_{j_r} \right)$$
$$+ \left(\sum_{j_1<j_2<\dots<j_{r-1}=1}^{p} k_{j_1} k_{j_2} \dots k_{j_{r-1}} \right) k_{p+1}$$
$$= \kappa_r^{(p)} + \kappa_{r-1}^{(p)} k_{p+1}$$

which completes the proof.

In this lemma $\kappa_r^{(p)}$ and $\kappa_r^{(p+1)}$ is not defined on the same manifold. For example $\kappa_r^{(p)}$ and $\kappa_r^{(p+1)}$ are defined on p-dimensional and (p+1)-dimensional manifolds, respectively. We know that p-dimensional manifold is included in the p+1-dimensional manifold. So we can apply the calculation rules to these functions at every point of the manifolds.

Now, we can give the following theorem.

Theorem 2.2. Let M and $\overline{M}$ be two hypersurfaces in E^{n+1}, such that $\overline{M}$ is parallel to M. For any natural numbers s, $1 \le s \le n$. $\kappa_s^{(n)}$ and $\overline{\kappa}_s^{(n)}$ denote the s-th higher order Gaussian curvatures of M and $\overline{M}$, respectively.

Then we have

$$\overline{\kappa}_s^{(n)} = \frac{\sum_{i=s}^{n} \frac{i(i-1)\ \dots\ (i-s+1)}{s!} r^{i-s} \kappa_i^{(n)}}{1 + \sum_{i=s}^{n} r^i \kappa_i^{(n)}}$$

Proof: We will apply the induction method to complete the proof. This method has two steps:

First step: we show that the theorem is true for the cases

(i) s = 1 and n = 1, n = 2, n = 3

(ii) s = 2 and n = 2, n = 3, n = 4

(iii) s = 3 and n = 3, n = 4, n = 5.

Second step: we assume that the theorem is true for s = q, n = p and then we will show that it is also true for s = q, n = p+1.

At first:

(i) For s = 1, n = 1, then we have

$$\overline{\kappa}_1^{(1)} = \sum_{j=1}^{1} \overline{k}_j = \frac{k}{1+rk_1} = \frac{\kappa_1^{(1)}}{1+r\kappa_1^{(1)}},$$

$$= \frac{\sum_{i=1}^{1} ir^i \kappa_i^{(1)}}{1 + \sum_{i=1}^{1} r^i \kappa_i^{(1)}}.$$

For s = 1, n = 2, then

$$\overline{\kappa}_1^{(2)} = \sum_{j=1}^{2} \overline{k}_j = \overline{k}_1 + \overline{k}_2 = \kappa_1^{(1)} + \overline{k}_2$$

$$= \frac{\sum_{i=1}^{2} ir^{i-1} \kappa_i^{(2)}}{1 + \sum_{i=1}^{2} r^i \kappa_i^{(2)}}$$

For s = 1, n = 3, then

$$\bar{\kappa}_1^{(3)} = \sum_{j=1}^{3} \bar{k}_j = \bar{k}_1 + \bar{k}_2 + \bar{k}_3 = \kappa_1^{(2)} + \bar{k}_3$$

$$= \frac{\sum_{i=1}^{3} ir^{i-1}\kappa_i^{(3)}}{1 + \sum_{i=1}^{3} r^i \kappa_i^{(3)}} .$$

(ii) For s = 2, n = 2, we have

$$\bar{\kappa}_2^{(2)} = \sum_{j_1<j_2=1}^{2} \bar{k}_{j_1} \bar{k}_{j_2} = \bar{k}_1 . \bar{k}_2 = \frac{\kappa_2^{(2)}}{1 + r\kappa_1^{(2)} + r^2\kappa_2^{(2)}}$$

$$= \frac{\sum_{i=2}^{2} \frac{i(i-1)}{2!} r^{i-2} \kappa_i^{(2)}}{1 + \sum_{i=1}^{2} r^i \kappa_i^{(2)}} .$$

For s = 2, n = 3, we have

$$\bar{\kappa}_2^{(3)} = \sum_{j_1<j_2=1}^{3} \bar{k}_{j_1} \bar{k}_{j_2} = \bar{k}_1 . \bar{k}_2 + \bar{k}_1 . \bar{k}_3 + \bar{k}_2 . \bar{k}_3$$

$$= \bar{\kappa}_2^{(2)} + \frac{k_3}{1 + rk_3} \bar{\kappa}_1^{(2)}$$

$$= \frac{\sum_{i=2}^{3} \frac{i(i-1)}{2!} r^{i-2} \kappa_i^{(3)}}{1 + \sum_{i=1}^{3} r^i \kappa_i^{(3)}} .$$

For s = 2, n = 4, we have

$$\bar{\kappa}_2^{(4)} = \sum_{j_1<j_2=1}^{4} \bar{k}_{j_1} \bar{k}_{j_2} = \bar{k}_1 . \bar{k}_2 + \bar{k}_1 . \bar{k}_3 + \bar{k}_1 . \bar{k}_4 + \bar{k}_2 . \bar{k}_3 + \bar{k}_2 . \bar{k}_4 + \bar{k}_3 . \bar{k}_4$$

$$= \bar{\kappa}_2^{(3)} + \frac{k_4}{1 + rk_4} \bar{\kappa}_1^{(3)}$$

$$= \frac{\sum_{i=2}^{4} \frac{i(1-1)}{2!} r^{i-2} \kappa_i^{(4)}}{1 + \sum_{i=1}^{4} r^i \kappa_i^{(4)}}$$

(iii) For s = 3, n = 3, we have

$$\bar{\kappa}_3^{(3)} = \sum_{j_1<j_2<j_3=1}^{3} \bar{k}_{j_1}\,\bar{k}_{j_2}\,\bar{k}_{j_3} = \bar{k}_1 \,.\, \bar{k}_2 \,.\, \bar{k}_3$$

$$= \frac{\kappa_3^{(3)}}{1 + r\kappa_1^{(3)} + r^2\kappa_2^{(3)} + r^3\kappa_3^{(3)}}$$

$$= \frac{\sum_{i=3}^{3} \frac{i(i-1)(i-2)}{3!} r^{i-3}\kappa_i^{(3)}}{1 + \sum_{i=1}^{3} r^i\kappa_i^{(3)}} .$$

For s = 3, n = 4, we have

$$\bar{\kappa}_3^{(4)} = \sum_{j_1<j_2<j_3=1}^{4} \bar{k}_{j_1}\,\bar{k}_{j_2}\,\bar{k}_{j_3} = \bar{k}_1.\bar{k}_2.\bar{k}_3 + \bar{k}_1.\bar{k}_3.\bar{k}_4 + \bar{k}_2.\bar{k}_3.\bar{k}_4 + \bar{k}_1.\bar{k}_2.\bar{k}_4$$

$$= \bar{\kappa}_3^{(3)} + \frac{k_4}{1 + rk_4}\,\bar{\kappa}_2^{(3)}$$

$$= \frac{\sum_{i=3}^{4} \frac{i(i-1)(i-2)}{3!} r^{i-3}\kappa_i^{(4)}}{1 + \sum_{i=1}^{4} r^i\kappa_i^{(4)}} .$$

For s = 3, n = 5, we have

$$\bar{\kappa}_3^{(5)} = \sum_{j_1<j_2<j_3=1}^{5} \bar{k}_{j_1}\,\bar{k}_{j_2}\,\bar{k}_{j_3} = \bar{k}_1.\bar{k}_2.\bar{k}_3 + \bar{k}_1.\bar{k}_2.\bar{k}_4 + \bar{k}_1.\bar{k}_2.\bar{k}_5 + \bar{k}_1.\bar{k}_3.\bar{k}_4 + \bar{k}_1.\bar{k}_3.\bar{k}_5$$

$$+ \bar{k}_1.\bar{k}_4.\bar{k}_5 + \bar{k}_2.\bar{k}_4.\bar{k}_5 + \bar{k}_2.\bar{k}_3.\bar{k}_4 + \bar{k}_2.\bar{k}_3.\bar{k}_5 + \bar{k}_3.\bar{k}_4.\bar{k}_5$$

$$= \bar{\kappa}_3^{(4)} + \frac{k_5}{1 + rk_5}\,\bar{\kappa}_2^{(4)}$$

$$= \frac{\sum_{i=3}^{5} \frac{i(i-1)(i-2)}{3!} r^{i-3}\kappa_i^{(5)}}{1 + \sum_{i=1}^{5} r^i\kappa_i^{(5)}} .$$

Second step:

Now, let assume that the theorem is true for s = q, n = p and we will show that it is also true for s = q, n = p+1.

This means that

$$\bar{\kappa}_q^{(p)} = \frac{\sum_{i=q}^{p} \frac{i(i-1)\ldots(i-q+1)}{q!} r^{i-q} \kappa_i^{(p)}}{1 + \sum_{i=1}^{p} r^i \kappa_i^{(p)}}$$

and we will show that

$$\bar{\kappa}_q^{(p+1)} = \frac{\sum_{i=q}^{p+1} \frac{i(i-1)\ldots(i-q+1)}{q!} r^{i-q} \kappa_i^{(p+1)}}{1 + \sum_{i=1}^{p+1} r^i \kappa_i^{(p+1)}} .$$

In the equation

$$\bar{\kappa}_q^{(p+1)} = \bar{\kappa}_q^{(p)} + \frac{k_{p+1}}{1 + rk_{p+1}} \bar{\kappa}_{q-1}^{(p)}$$

we can write that

$$\bar{\kappa}_q^{(p+1)} = \frac{\sum_{i=q}^{p} \frac{i(i-1)\ldots(i-q+1)}{q!} r^{i-q} \kappa_i^{(p)}}{1 + \sum_{i=1}^{p} r^i \kappa_i^{(p)}} + \frac{k_{p+1}}{1 + rk_{p+1}} \left[\frac{\sum_{i=q-1}^{p} \frac{i(i-1)\ldots(i-q+2)}{(q-1)!} r^{i-q+1} \kappa_i^{(p)}}{1 + \sum_{i=1}^{p} r^i \kappa_i^{(p)}} \right]$$

$$= \frac{\left(1 + rk_{p+1}\right) \sum_{i=q}^{p} \frac{i(i-1)\ldots(i-q+1)}{q!} r^{i-q} \kappa_i^{(p)} + k_{p+1} \sum_{i=q-1}^{p} \frac{i(i-1)\ldots(i-q+2)}{(q-1)!} r^{i-q+1} \kappa_i^{(p)}}{\left(1 + rk_{p+1}\right) \left[1 + \sum_{i=1}^{p} r^i \kappa_i^{(p)} \right]}$$

$$= \frac{\kappa_q^{(p+1)} + \sum_{i=q}^{p-1} \frac{(i+1)i(i-1)\ldots(i-q+2)}{q!} r^{i-q+1} \kappa_{i+1}^{(p+1)} + \frac{(p+1)p(p-1)\ldots(p-q+2)}{q!} r^{p-q+1} \kappa_{p+1}^{(P+1)}}{1 + r\kappa_1^{(p+1)} + \sum_{i=1}^{p-1} r^{i-1} \kappa_{i+1}^{(p+1)} + r^{p+1} \kappa_{p+1}^{(p+1)}} .$$

Then

$$\bar{\kappa}_q^{(p+1)} = \frac{\sum_{i=q}^{p+1} \frac{i(i-1)\ldots(i-q+1)}{q!} r^{i-q} \kappa_i^{(p+1)}}{1 + \sum_{i=1}^{p+1} r^i \kappa_i^{(p+1)}}$$

which completes the proof in this case.

Now; we can put q = s and p+1 = n in the last equation, we obtain

$$\kappa_s^{(n)} = \frac{\sum_{i=s}^{n} \frac{i(i-1)\ldots(i-s+1)}{s!} r^{i-s} \kappa_i^{(n)}}{1 + \sum_{i=1}^{n} r^i \kappa_i^{(n)}}$$

which completes the proof of the theorem.

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