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# THE REPRESENTATION OF SERIES-PARALLEL-ORDERED SETS

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**ABSTRACT**

In [6] it is shown, that weak orders, a subclass of series prallel posets, are represented by bands. In this paper a representation of series parallel posets is given and it is shown how all weak ordered finite bands can be constructed. We first want to give a construction of CDC's as a set of special n-tuppels of natural numbers. After this we assign to each of these tuppels a rectangular band and show how weak ordered bands can be thus constructed. Moreover all weak ordered bands are constructed in this way.

## 1. INTRODUCTION

In 1986 Mitsch [3] showed that to any semigroup S a natural partial order can be defined by

$$a \leq b \text{ if } a = bx = yb = ax \text{ for some } x, y \in S^1.$$

This order is an extension of the natural partial order on idepotent elements. In [4] Neggers showed that posets and poset homomorphisms form a category which is equivalent to the category of pogroupoids. This idea was carried on in [2], where it is shown that a pogroupoid of a weak order is a semigroup.

Therefore the following question seemed natural: When is a poset a natural poset of a semigroup? We also say that a semigroup represents a poset if the natural poset of the semigroup is isomorphic with the given poset. Hence we can reformulate the above question: When is a poset represented by a semigroup? Some classes are known to be represented, which we want to introduce in this paper.

We use $a < b$ to express $a \leq b$ but $a \neq b$. And if $a \nleq b$ and $b \nleq a$ then we write $a \parallel b$. A poset is called a weak order if $\parallel$ defines an equivalence relation.

We need the equivalence J, one of Green's relation, given by

a J b if $S^1aS^1 = S^1bS^1$.

On the bands we have a J b if a = aba and b = bab [1]. J is used in theorem 4.1 and in the proof of theorem 4.2 and theorem 4.3.

## 2. SERIES PARALLEL POSETS

A partial ordered set is called series parallel if it can be constructed from singeltons using the operations of disjoint sum, denoted by '+', and linear sums, denoted by '⊕'. For example trees are seriesparallel as well as weak orders. A known result is:

**Theorem 2.1. (Series-parallel-N-Theorem)** [5] [9] A finite ordered set is series parallel if and only if it contains no subset isomorphic to N.

To prove the maintheorem we need results found in [7]. Special bands are used there, which are defined as follows:

**Definition 2.2.** A band (respectively a semigroup) is called a RZ band (respectively a RZ semigroup) if its set of minimal elements form a rightzero semigroup.

**Theorem 2.3.** [7] Let $\mathcal{O}_i$, i = 1, 2 be orders which are represented by RZ bands, then $\mathcal{O}_1 + \mathcal{O}_2$ is represented by $(B_1 \cup B_2, *)$. The multiplication is given by

$$x * y = \begin{cases} m_i y & \text{if } x \notin B_i,\ y \in B_i \\ xy & \text{else} \end{cases}$$

where $m_i$ is a fixed minimal element in $B_i$.

**Lemma 2.4.** [7] Let $\mathcal{O}_i$, i = 1, 2 two orders which are represented by bands $B_i$ , i = 1, 2 then $\mathcal{O}_1 \oplus \mathcal{O}_2$ is represented by $(B_1 \cup B_2, *)$ where

$$x * y = \begin{cases} x & \text{if } x \in B_1, y \in B_2 \\ y & \text{if } x \in B_2, y \in B_1 \\ xy & \text{else} \end{cases}$$

These results show that if two posets are represented by RZ bands, than also their disjoint and linear sum is represented by a RZ band. Consequently:

**Theorem 2.5.** Any series parallel order $\mathcal{O}$ is represented by a RZ band B.

**Proof.** Let **a** be the defining expression of $\mathcal{O}$. If $\mathcal{O}$ is not a singelton the expression consists of two smaller subexpressions, that are connected by either '+' or '⊕'. Since we showed that a cardinal sum as well as a linear sum of orders, which are represented by RZ bands are again represented by RZ bands and obviously the singelton is represented by a RZ band, an easy induction argument on the length of the expression **a** completes in proof.

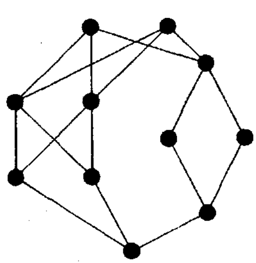
**Example 2.6.** Let **a** be the defining expression of a series parallel ordered set $\mathcal{O}$, given by

$$\mathbf{a} = (1 \oplus (((1+1) \oplus (1 + 1)) + (1 \oplus (1+1) \oplus 1))) \oplus (1+1)$$

To distinguish between the elements we rewrite **a** as:

$$\mathbf{a} = (e_1 \oplus \underbrace{(((e_2+e_3) \oplus (e_4+e_5))}_{P_1} + \underbrace{(e_6 \oplus (e_7+e_8) \oplus e_9)))}_{P_2} \oplus \underbrace{(e_{10}+e_{11})}_{P_3}$$

The Hasse diagram of this order is:



Expression of the form $e_i + e_j$ are represented by bands with the following multiplication table:

| * | $e_i$ | $e_j$ |
|---|---|---|
| $e_i$ | $e_i$ | $e_j$ |
| $e_j$ | $e_i$ | $e_j$ |

Using this for $e_{10}$ and $e_{11}$ we get the table for $P_3$. The tables for $P_1$ and $P_2$ are combinations of this table as described above. We get

| * | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|---|---|---|---|---|
| $e_1$ | $e_2$ | $e_3$ | $e_2$ | $e_2$ |
| $e_2$ | $e_2$ | $e_3$ | $e_3$ | $e_3$ |
| $e_3$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
| $e_4$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |

| * | $e_6$ | $e_7$ | $e_8$ | $e_9$ |
|---|---|---|---|---|
| $e_6$ | $e_6$ | $e_6$ | $e_6$ | $e_6$ |
| $e_7$ | $e_6$ | $e_7$ | $e_8$ | $e_7$ |
| $e_8$ | $e_6$ | $e_7$ | $e_8$ | $e_8$ |
| $e_9$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ |

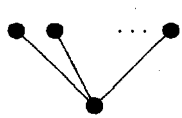
Finally the complete table is given by:

| * | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $e_1$ | $e_1$ | $e_1$ | $e_1$ | $e_1$ | $e_1$ | $e_1$ | $e_1$ | $e_1$ | $e_1$ | $e_1$ |
| $e_2$ | $e_1$ | $e_2$ | $e_3$ | $e_2$ | $e_2$ | $e_6$ | $e_6$ | $e_6$ | $e_6$ | $e_2$ | $e_2$ |
| $e_3$ | $e_1$ | $e_2$ | $e_3$ | $e_3$ | $e_3$ | $e_6$ | $e_6$ | $e_6$ | $e_6$ | $e_3$ | $e_3$ |
| $e_4$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_6$ | $e_6$ | $e_6$ | $e_4$ | $e_4$ |
| $e_5$ | $e_1$ | $e_2$ | $e_3$ | $e_2$ | $e_5$ | $e_6$ | $e_6$ | $e_6$ | $e_6$ | $e_5$ | $e_5$ |
| $e_6$ | $e_1$ | $e_2$ | $e_3$ | $e_2$ | $e_2$ | $e_6$ | $e_7$ | $e_6$ | $e_6$ | $e_6$ | $e_6$ |
| $e_7$ | $e_1$ | $e_2$ | $e_3$ | $e_2$ | $e_2$ | $e_6$ | $e_7$ | $e_8$ | $e_7$ | $e_7$ | $e_7$ |
| $e_8$ | $e_1$ | $e_2$ | $e_3$ | $e_2$ | $e_2$ | $e_6$ | $e_7$ | $e_8$ | $e_8$ | $e_8$ | $e_8$ |
| $e_9$ | $e_1$ | $e_2$ | $e_3$ | $e_2$ | $e_2$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_9$ | $e_9$ |
| $e_{10}$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ |
| $e_{11}$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ |

## 3. CROWN DIAMONDS CHAINS

We recall some definitions and a result found in [6]. Some finite semilattice have a special form and are obviously weakly ordered.

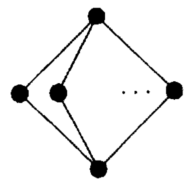
**Definition 3.1.** A semilattice Y of the form



A Crown

is called a crown.

**Definition 3.2.** A semilattice Y of the form



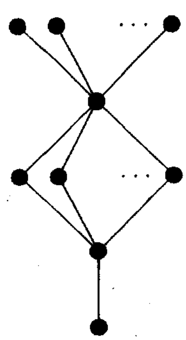
A Diamond

is called a diamond.

Let $\wp_i$, i = 1, 2 be two posets such that has $\wp_i$ a greatest element g and $\wp_2$ has a smallest element s. Then the glue-linear sum of $\wp_1$ and $\wp_2$ is defined to be

$$\wp_1 \oplus (\wp_2 \setminus \{s\}) = (\wp_1 \setminus \{g\}) \oplus \wp_2.$$

**Definition 3.3.** A semilattice Y is called a crown-diamond-chain if it is glue-linear sum of chains, diamonds and a crown as last summand or a glue-linear sum of chains and diamonds.



A Crown-Diamond-Chain

The following theorem gives a description of finite weakly ordered semilattices:

**Theorem 3.4.** [6] A finite semilattice Y is weak-ordered if and only if it is a crown-diamond-chain.

Since a CDC is weak ordered it is also a series parallel order. The defining expression of a CDC has the following obvious properties:

1. it doesn't start with (1 + ... + 1) and

2. it contains no subpart $\underbrace{(1 + ... + 1)}_{k_1} \oplus \underbrace{(1 + ... + 1)}_{k_2}$, $k_1, k_2 > 1$.

Consequently and CDC can be represented by a sequence

$$c = (x_1\ x_2, ...\ x_n) \text{ with } \begin{cases} x_i > 0 \text{ if } i \text{ is odd} \\ x_i > 1 \text{ if } i \text{ is even.} \end{cases}$$

Such a seqence is transformed into a defining expression as follows:

$$\mathbf{a} = \underbrace{(1 \oplus ... \oplus 1)}_{x_1} \oplus \underbrace{(1 + ... + 1)}_{x_2} \oplus \underbrace{(1 \oplus ... \oplus 1)}_{x_3} \oplus ...$$

From this it should be clear how to receive a sequence from a defining expression.

But these sequences do not only describe the general structure of a CDC. With its help a set of n-tuppels can be given, such that each element of the given CDC corresponds to one of these tuppels and this set can be endowed with a multiplication, that yields a CDC structure.

**Definition 3.5.** Let $c = (x_1, ..., x_n)$ be a sequence that describes a CDC, then $\langle c \rangle$ denotes the set of all n-tuppels $(t_1, ..., t_n)$ such that $0 \leq t_i \leq x_i$ and if $t_i < x_i$ than $t_{i+1} = 0$ and $\max (t_i : i = 1, ..., n) > 0$.

**Lemma 3.6.** Let $c = (x_1, ..., x_n)$ then $|\langle c \rangle| = \sum_{i=1}^{n} x_i$.

**Proof.** We prove this by induction. Let the length of c be one, then the result follows immediatly. Now we assume that for sequences c with length k the result is true. If we take a sequence with length k+1, then we have $\sum_{i=1}^{n} x_i$ elements with $t_{k+1} = 0$. Adding the $x_{k+1}$ cases for $tk > 0$ we get the result. Note that if $t_{k+1} > 0$ then $t_i = x_i$ for $i \leq k$.

Now that we saw that $\langle c \rangle$ has exactly the same number than the CDC which is defined by c, we will give a multiplication on $\langle c \rangle$ such that it becomes a CDC with the same defining expression, that means $\langle c \rangle$ represents the given CDC and at the same time we saw, that all CDC can be constructed as a set $\langle c \rangle$ for some, one, sequence c.

**Theorem 3.7.** Let $c = (x_1, ..., x_n)$. If we define on $\langle c \rangle$ the following multiplication:

$$(s_1, ..., s_n) * (t_1, ..., t_n) = ([s_1, t_1], ..., [s_n, t_n])$$

where

$$[s_i,t_i] := \begin{cases} 0 \text{ if } i \text{ is even and } s_i \neq t_i \\ \min(s_i,t_i) \text{ else} \end{cases}$$

then $(\langle c \rangle, *)$ becomes a CDC.

**Proof.** The given multiplication is idempotent and commuttive. We have to show that it is associative and closed. It sufficies to show hat [.,.] is associative on the components. We observe the case where $i = 2k$ and $i \neq 2k$.

$$i \neq 2k \quad [[a_i,b_i],d_i] = \min(\min(a_i,b_i),d_i) = \min(a_i,b_i,d_i) = [a_i,[b_i,d_i]]$$

Now we turn to the even components:

$$[[a_i, b_i], d_i] := \begin{cases} [a_i, d_i] \text{ if } a_i = b_i \\ [0, d_i] = 0 \text{ if } a_i \neq b_i \end{cases} \qquad [a_i, [b_i, d_i]] := \begin{cases} [a_i, d_i] \text{ if } b_i = d_i \\ [a_i, 0] = 0 \text{ if } b_i \neq d_i \end{cases}$$

To check whether these products are equal, we have to consider two cases $a_i = d_i$ and $a_i \neq d_i$.

If $a_i = d_i$ and $a_i = b_i$ then $b_i = d_i$ too and $[[a_i,b_i],d_i] = a_i = [a_i, [b_i,d_i]]$. If $a_i = d_i$ and $a_i \neq b_i$ then $d_i \neq b_i$ too and we have $[[a_i,b_i],d_i] = [0,d_i] = 0$ $[a_i,0] = [a_i,[b_i,d_i]]$. Now we look after $a_i$ v $d_i$. In this case all of the above outcomes ore 0. Consequently the multiplication * is associative since:

$$\begin{aligned}(a*b)*d &= \left([[a_1,b_1],d_1],...,[[a_n,b_n],d_n]\right) = \\ &= \left([a_1,[b_1,d_1]],...,[a_n,[b_n,d_n]]\right) = \\ &= a*(b*d)\end{aligned}$$

It remains to show that the multiplication is closed. If $a = b$ than we know that $a*b = a*a = a$. Let $a \neq b$, then there is an index i, minimal, such that $a_i \neq b_i$, say $a_i < b_i$. Conseqeuently $a_i < x_i$ and $a_{i+1} =$

0. Moreover $a_k = 0$ when $k > i$. All components $d_k = [a_k,b_k]$ of $a*b$ with $k > i$ are 0 and the components $d_k$ with $k < i$ are $a_k = b_k$. If i is even then $d_i = 0$ and $a*b \in \langle c\rangle$. If i is odd then $d_i = \min(a_i,b_i) = a_i$ and $a*b$ is in $\langle c\rangle$.

This semilattice is obviously a CDC.

## 4. WEAK ORDERED BANDS

Now we assign to each element of a CDC = $(\langle c\rangle,*)$ a rectangular band $RB_a$, $a \in \langle c\rangle$ . The rectangular bands $RB_a$ are arbitrary except for the elements $a = (x_1, x_2, ..., x_{2k+1}, 0, ..., 0)$. For these elements $RB_a$ consists of only one element, say $x_a$. On the set

$$W := \{(a,x) : a \in \langle c\rangle, x \in RB_a\}$$

we define the following multiplication.

$$(a,x) \odot (b,y) = (a * b\ \theta(a, b, x, y))$$

where $\theta(a, b, x, y)$ is defined by

$$\theta(a, b, x, y) := \begin{cases} x \text{ if } a = a * b,\ a \neq b \\ y \text{ if } b = a * b,\ a \neq b \\ xy \text{ if } a = b \\ x_{a*b} \text{ else} \end{cases}$$

Here $x_{a*b}$ is the only element in $RB_{a*b}$.

We need the following

**Theorem 4.1.** [6] A finite band B is weak ordered if and only if the following properties hold:

1. B/J is a crown-diamond-chain and

2. $a < b \Leftrightarrow aJ < bJ$

to show that

**Theorem 4.2.** $(W,\odot)$ is a weak ordered band.

**Proof.** The given multiplication is obvioysly closed and it is easy to see, that it is idempotent, since

$$(a,x) \odot (a,x) = (a*a, \theta(a, a, x, x)) = (a,x)$$

Now we show that $\odot$ is a associative.

$$\begin{aligned}&((s_1,x_1) \odot (s_2,x_2)) \odot (s_3,x_3) = \\ &= (s_1 * s_2, \theta(s_1, s_2, x_1, x_2)) \odot (s_3,x_3) = \\ &= ((s_* s_2) * s_3, \theta(s_1 * s_2, s_3, \theta(s_1, s_2, x_1, x_2), x_3))\end{aligned} \quad (1)$$

$$\begin{aligned}&(s_1,x_1) \odot ((s_2,x_2) \odot (s_3,x_3)) = \\ &= (s_1,x_1) \odot (s_2 * s_3, \theta(s_2, s_3, x_2, x_3)) = \\ &= (s_* (s_2 * s_3), \theta(s_1, s_2 * s_3, x_1, \theta(s_2, s_3, x_2, x_3)))\end{aligned} \quad (2)$$

So the multiplication is associative if(1) = (2). The left side, this is (1) depends on $\theta(s_1, s_2, x_1, x_2)$ hence we get

**L1** $\theta(s_1, s_3, x_1, x_3)$ if $s_1 < s_2$

**L2** $\theta(s_2, s_3, x_2, x_3)$ if $s_2 < s_1$

**L3** $\theta(\underbrace{s_1}_{=s_2}, s_3, x_1, x_2x_3)$ if $s_2 = s_3$

**L4** $\theta(s_1 * s_2, s_3, x_{s_1 * s_2}, x_3)$ if $s_1 \parallel s_2$

The right side depend on $\theta(s_2, s_3, x_2, x_3)$ and we get

**R1** $\theta(s_1, s_2, x_1, x_2)$ if $s_2 < s_3$

**R2** $\theta(s_1, s_3, x_1, x_3)$ if $s_3 < s_2$

**R3** $\theta(s_1, \underbrace{s_2}_{=s_3}, x_1, x_2x_3)$ if $s_2 = s_3$

**R4** $\theta(s_1, s_2 * s_3, x_1, x_{s_2 * s_3})$ if $s_2 \parallel s_3$

To check associativity we have to show that $Li = Rj \ \forall i,j$.

**L1R1** $s_1 < s_2$ and $s_2 < s_3$ hence $s_1 < s_3$ and $L1 = x_1 = R1$.

**L1R2** L1 equals R2 indeed.

**L1R3** $s_1 < s_2$ and $s_2 = s_3$ yields $s_1 < s_3$ hence $L1 = x_1 = R3$.

**L1R4** $s_1 < s_2$ and $s_2 \parallel s_3$ yields $s_1 < s_3$ since $\langle c\rangle$ is a CDC, hence L1 = $x_1$ = R4.

**L2R1** $s_2 < s_1$ used in R1 gives $x_2$ and $s_2 < s_3$ yields $x_2$ in L2.

**L2R2** $s_2 < s_1$ and $s_3 < s_2$ yields $s_3 < s_1$ hence L2 = $x_3$ = R2.

**L2R3** $s_2 < s_1$ and $s_3 = s_2$. Hence $s_3 < s_1$ and L2 = $x_2x_3$ = R3.

**L2R4** Since $s_2 \parallel s_3$ and because $\langle c\rangle$ is a CDC we have also $s_3 < s_1$. Consequently $s_2 * s_3 < s_3 < s_1$ and L2 = $x_{s_2 * s_3}$ = R4.

**L3R1** $s_1 = s_2$ and $s_2 < s_3$ yields $s_1 < s_3$ hence L3 = $x_1x_2$ = R1.

**L3R2** $s_1 = s_2$ and $s_3 < s_2$ yields $s_3 < s_1$ hence L3 = $x_3$ = R2.

**L3R3** $s_1 = s_2$ and $s_2 = s_3$ hence $L_3 = (x_1x_1)x_3 = x_1(x_2x_3)$ = R3.

**L3R4** $s_1 = s_2$ and $s_2 \parallel s_3$ yields also $s_1 \parallel s_3$ hence L3 = $x_{s_1 * s_3}$ = $x_{s_2 * s_3}$ = R4, since $s_2 * s_3 < s_2 = s_1$.

**L4R1** $s_1 \parallel s_2$ and $s_2 < s_3$ yields $s_1 < s_3$ since $\langle c\rangle$ is a CDC. Hence L4 = $x_{s_1 * s_2}$ = R1 because $s_1 * s_2 < s_2 < s_3$.

**L4R2** Here we get $s_3 < s_1$ and consequently $s_3 \leq s_1 * s_2$. Hence L4 = $x_3$ = R2.

**L4R3** We have $s_1 \parallel s_2$ and $s_2 = s_3$ therefore $s_1 \parallel s_3$ and L4 = $x_{s_1 * s_2}$ = R3 since $s_1 * s_2 < s_2 = s_3$.

**L4R4** $s_1 \parallel s_2$ and $s_2 \parallel s_3$ yields $s_1 \parallel s_3$ since $\langle c\rangle$ is a CDC hence $s_1 * s_2 = s_2 * s_3 < s_1, s_3$. Consequently L4 = $x_{s_1 * s_2}$ = $x_{s_2 * s_3}$ = R4.

This proves that $(W,\odot)$ is a band. It is clear that $W/J \cong \langle c\rangle$ and $(a,x) < (b,y)$ if and only if $a < b$, hence $(W,\odot)$ is a weak ordered band according to theorem 4.1.

**Theorem 4.3.** Let B be a weak ordered band, then there are a CDC $(\langle c\rangle,*)$, rectangular bands $RB_a$, $a \in \langle c\rangle$, such that

$$B \cong (W,\odot)$$

where $(W,\odot)$ is defined as shown above.

**Proof.** Since B is weak ordered we know that B/J is a CDC and B is a CDC of rectangular bands.

Now if $e \in RB_a$ and $f \in RB_b$ than $ef \in RB_{a*b}$, where '*' denotes the multiplication in B/J. More precisely if $a < b$ then $ef = e = fe$, since $e < f$ if $a < b$. Now suppose that $a \parallel b$. Then we know that $ef < e$ and $ef < f$ since $a*b < a$ and $a * b < b$. Consequently $ef = efe = fe$. Moreover let $e' \in RB_a$ and $f' \in RB_b$ then

$$e'f' = e'ee'f' = e' \underbrace{ef'}_{\in RB_{a*b}} e' = ef' = ef'ff' = ef.$$

But we can even show more. Let $p \in RB_{a*b}$ then

$(ef)p = p$ since $p < e$ and $p < f$

we also have $p(ef) = p$ and therefore

$ef = (ef)p(ef) = p$

and $RB_{a*b}$ consists only of one element.

These observations showed that $B \cong (W,\leq)$ where

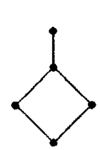
$W = \{(a,x) : a \in B/J,\ x \in RB_a\}$

Hence we established the required statement.

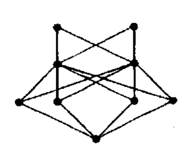
**Example 4.4.** Let $c = (1, 2, 2)$ be a CDC with

$\langle c\rangle = \{(1, 0, 0), (1, 1, 0), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$.

This CDC is given by the following diagram



$c = (1,2,2)$



$(W,\leq)$

We choose isomorph rectangular bands $RB_a$ except for a = (1, 0, 0) what must be a oneelementic set. $RB_a = \{x,y\}$ with xy = yy = y and yx = xx = x then we get

$$
\begin{aligned}
W = \{&((1, 0, 0), 1),\\
&((1, 1, 0)\ x), ((1, 1, 0), y), ((1, 2, 0), x), ((1, 2, 0), y),\\
&((1, 2, 1), x), ((1, 2, 1), y),\\
&((1, 2, 2), x), ((1, 2, 2), y)\} =\\
= \{&w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}
\end{aligned}
$$

The natural partial order i given above. The multiplication table is:

| * | $w_0$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | $w_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $w_0$ | $w_0$ | $w_0$ | $w_0$ | $w_0$ | $w_0$ | $w_0$ | $w_0$ | $w_0$ | $w_0$ |
| $w_1$ | $w_0$ | $w_1$ | $w_2$ | $w_0$ | $w_0$ | $w_1$ | $w_1$ | $w_1$ | $w_1$ |
| $w_2$ | $w_0$ | $w_1$ | $w_2$ | $w_0$ | $w_0$ | $w_2$ | $w_2$ | $w_2$ | $w_2$ |
| $w_3$ | $w_0$ | $w_0$ | $w_0$ | $w_3$ | $w_4$ | $w_3$ | $w_3$ | $w_3$ | $w_3$ |
| $w_4$ | $w_0$ | $w_0$ | $w_0$ | $w_3$ | $w_4$ | $w_4$ | $w_4$ | $w_4$ | $w_4$ |
| $w_5$ | $w_0$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_5$ | $w_5$ |
| $w_6$ | $w_0$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_6$ | $w_6$ |
| $w_7$ | $w_0$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | $w_8$ |
| $w_8$ | $w_0$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | $w_8$ |

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