Commun. Fac. Sci. Univ. Ank. Series A1
V. 47. pp. 73-83 (1998)

ISOMETRIES OF TAXICAB GEOMETRY

İ. KOCAYUSUFOĞLU* and E. ÖZDAMAR**

* *Department of Mathematics, Osmangazi University, Eskişehir, TURKEY*
** *Department of Mathematics, Uludağ University, Bursa, TURKEY*

(Received March 2, 1998; Accepted May 18, 1998)

ABSTRACT

The taxicab metric, of course, was known before the taxicab geometry was introduced in 1975. Since then, the studies have shown that the taxicab geometry is better model in urban world. The definition of inner-product and norm in taxicab geometry are given in [1]. In this paper, we will discuss some properties of taxicab norm and the isometries of taxicab geometry.

1. INTRODUCTION

Although we can define various metrics on the plane R^2, we can state the most common three of them as follows: Given $A = (x_1, y_1)$, $B = (x_2, y_2)$ in R^2,

$$d_E(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$d_M(A, B) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$
$$d_T(A, B) = |x_1 - x_2| + |y_1 - y_2|.$$

The first one is known universally and named by Euclidean metric, second one is known as the maximum metric. The third metric arises in the problem of travel within a city which has a set of parallel roads which intersects a set of parallel avenues at right angles.

By using this metric, named by taxicab metric, E.F. Krause [3] has defined a new geometry, taxicab geometry. He mentioned in his book, Taxicab Geometry, that the taxicab geometry is a non-Euclidean geometry. It has the same coordinate plane as far as the points andlines are concerned. Only the distance function is different. Intuitively the taxicab distance from a point (a_1, a_2) to a point (b_1, b_2) is suggested by the route a taxicab might use. It satisfies all thirteen axioms of Euclid, except one, the side-angle-side axioms [5].

There are, of course, some differences between the taxicab and Euclid geometry which is studied in [2]. Basically, the graphs are different, and while $\pi_E = 3{,}14$ in Euclid geometry, $\pi_T = 4$ in taxicab geometry. The inner-product and the norm in taxicab geometry is defined in [1]. We will recall these definitions here and then we will discuss some properties of norm and finally we will describe the isometries of taxicab geometry.

2. SOME PROPERTIES OF TAXICAB NORM.

Let us denote the taxicab-plane by $R^2_T = (R^2, d_T)$.

Definition 1. Given $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$ in R^2_T, we define the taxicab inner-product by

$$<\alpha, \beta>_T = \varepsilon_1|a_1b_1| + \varepsilon_2|a_2b_2|$$

$$\text{where } \varepsilon_i = \begin{cases} 1, & a_ib_i > 0 \\ -1, & a_ib_i < 0 \end{cases}, \ i = 1, 2.$$

Definition 2. Given $\alpha = (a_1, a_2)$ in R^2_T, we define the taxicab norm of α by

$$\|\alpha\|_T = \sqrt{<\alpha, \alpha>_T + 2\,|a_1a_2|}.$$

It is clear that $\|\alpha\|_T = d_T(0, \alpha)$.

As it is well known, every inner-product induces a metric, but the inverse is not true. Namely, every metric may not be induced by an inner-product [4]. We shall prove that the taxicab metric can not be induced by any inner-product.

Theorem 1. The taxicab metric, d_T, can not be induced by any inner-product.

Proof. We know that if the norm were induced by an inner-product, $<,>$, it would hold the polarization identity,

$$<\alpha, \beta> = \frac{1}{4}\left(\|\alpha + \beta\|^2 - \|\alpha - \beta\|^2\right).$$

So, for the taxicab norm, it should be true that

$$<\alpha, \beta> = \frac{1}{4}\left(\|\alpha + \beta\|_T^2 - \|\alpha - \beta\|_T^2\right).$$

It is straightforward computation that for standard vectors, $e_1 = (1,0)$ and $e_2 = (0, 1)$,

$$<e_1, e_1> = \frac{1}{4}\left(\|(2, 0)\|_T^2 - \|(0, 0)\|_T^2\right) = 1 ,$$

$$<e_2, e_2> = \frac{1}{4}\left(\|(0, 2)\|_T^2 - \|(0, 0)\|_T^2\right) = 1 ,$$

$$<e_1, e_2> = \frac{1}{4}\left(\|(1, 1)\|_T^2 - \|(1, -1)\|_T^2\right) = 0 ,$$

$$<e_2, e_1> = \frac{1}{4}\left(\|(1, 1)\|_T^2 - \|(-1, 1)\|_T^2\right) = 0 .$$

For $\alpha = (0, 1) = e_2$, and $\beta = (1, 2) = e_1 + 2\, e_2$, while

$$<\alpha, \beta> = <e_2, e_1 + 2e_2> = 2,$$

it is true, on the other hand, that

$$<\alpha, \beta> = \frac{1}{4}\left(\|(1, 3)\|^2 - \|(-1, -1)\|^2\right) = 3$$

which implies that it does not hold the polarization identity.

3. ISOMETRIES OF TAXICAB GEOMETRY

Once we have a metric, we may ask the question about the isometries which preserves the distance. We shall prove that the isometries of taxicab metric are all translations, T(2), and the orthogonal group $O_T(2)$, consisting of four reflections and four rotations defined below.

3.1. Translations

Let $T_a : R^2_T \to R^2_T$ such that $T_a(P) = a + P$ be translation function as in Euclidean plane R^2. For $P = (p_1, p_2)$, $Q = (q_1, q_2) \in R^2_T$, we have

$$\begin{aligned} d_T(T_a(P), T_a(Q)) &= d_T(a + P, a + Q) \\ &= d_T(P, Q) . \end{aligned}$$

So, T_a is an isometry. Thus, we proved the following theorem:

Theorem 2. All translations are isometries in taxicab geometry.

3.2. REFLECTIONS

We have only four reflections, S, that preserves the distance. Namely,

$S = \{(x, y) \mid x = 0, y = 0, y = x, \text{ and } y = -x\}$.

To see this we give the following theorems.

Theorem 3. Let P = (a, b). A point on a line y = mx, m ≠ 1, with minimum taxicab distance from P, is either B = (b/m, b) or C = (a, ma). For m = 1, any point on the line segment [BC] has the minimum taxicab distance from P (Figure 1).

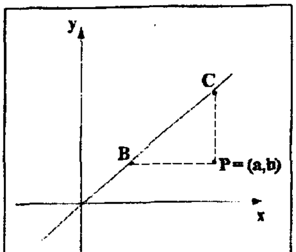

Figure 1

Let us define the reflection by a line in R^2_T. In Euclidean geometry, we found the reflection of a point A by a line *l*, by drawing an orthogonal line [AH], H ∈ *l*, from A to *l* and then in other side of the line *l*, with the same Euclidean distance on the line [AH], we get A′, the reflection of A. We define the reflection of a point in R^2_T as follows:

Definition 3. Consider the point C with minimum taxicab distance from P to the line *l*, and draw the line [PC]. Then choose the point P′ in the opposite side of the line *l* with respect to P such that $d_T(P, C) = d_T(P', C)$. The point P′ is called the reflection of P.

It is easy to show that all reflections in taxicab plane don't preserve the taxicab distance. In fact, we shall prove that the only reflections, preserving the taxicab distance are the reflections S.

Theorem 4. Let y = mx, m ≠ ±1. Then, a reflection by the line y = mx is an isometry iff m = 0 or m → ∞.

Proof. Since $f : R^2_T \to R^2_T$ is a reflection by the line y = mx, f can be defined by

$$f(x, y) = \begin{cases} (x, 2mx - y & , \quad 0 \le m < 1 \\ \left(\dfrac{2y - mx}{m}, y\right) & , \quad m > 1 \end{cases}$$

Suppose, now, f is an isometry. Then,

$d_T((x, y), (a, b)) = d_T(f(x, y), f(a, b))$.

For $0 \le m < 1$,

$|x - a| + |y - b| = |x - a| + |2ma - b - 2mx + y|$

which implies $m = 0$.

For $m > 1$,

$$|x - a| + |x - b| = \left|\frac{2b - ma}{m} - \frac{2y - mx}{m}\right| + |y - b|$$

which implies $m \to \infty$.

Corollary 5. The isometries of reflection by a line with the slope $m \neq \pm 1$, are the reflections by the lines $x = 0$ and $y = 0$.

Now, we claim that the other isometric reflections are the reflections by the lines $y = x$ and $y = -x$.

Given a point P, there are more than one points with minimum taxicab distance from P to the line $y = x$. Thus, the reflection by the line $y = x$ is well defined for each of such points. A horizontal line $y = b$, passing through the point $P = (a, b)$ intersects the line $y = x$ at the point $C = (b, b)$, and the vertical line intersects, it at $B = (a, a)$ (Figure 2).

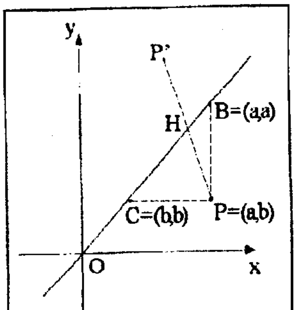

Figure2

Any point on [BC] has the minimum taxicab distance to the point P, with a distance $|b - a|$. Consider, now, a point $H = (u, u) \in [BC]$. For $0 \le \lambda \le 1$, we can write $u = b + \lambda(a - b)$. On the line [PH], for a point P such that $d_T(P, H) = d_T(P', H)$, we can also write

$f(P) = P' = (-a + 2b + 2\lambda(a - b), b + 2\lambda(a - b))$.

It is clear that

$$
\begin{aligned}
d_T(P', H) &= dT((-a + 2b + 2\lambda(a - b), b + 2\lambda(a - b)), (u, u)) \\
&= |-a + 2b + 2\lambda(a - b) - (b + \lambda(a - b))| + \\
&\quad |b + 2\lambda(a - b) - (b + \lambda(a - b))| \\
&= |-b + a - \lambda(a - b)| + |-\lambda(a - b)| \\
&= |1 - \lambda|\, |a - b| + |-\lambda|\, |(a - b)| \\
&= (1 - \lambda)\, |a - b| + \lambda|a - b| \quad ; 0 \le \lambda \le 1 \\
&= |a - b|
\end{aligned}
$$

Thus, P' is the reflection of P with respect to H.

Notation: We will denote such a reflection by H_λ.

Theorem 6. H_λ - reflection by the line y = x is an isometry iff $\lambda = \frac{1}{2}$.

Proof. Suppose an H_λ - reflection $f : R^2_T \to R^2_T$ is an isometry. Since

$$f(x, y) = (-x + 2y + 2\lambda(x - y), y + 2\lambda(x - y)),$$

we have

$$d_T(f(x, y), f(a, b)) = |-a + 2b + 2\lambda (a - b) - (-x + 2y + 2\lambda(x - y))| + |b + 2\lambda(a - b) - (y + 2\lambda(x - y))|.$$

On the other hand,

$$d_T((x, y), (a, b)) = |x - a| + |y - b|.$$

Thus, solving the equation

$$d_T(f(x, y), f(a, b)) = d_T((x, y), (a, b))$$

we get $\lambda = \frac{1}{2}$, as claimed. Clearly, the converse is also true.

Corollary 7. An isometric reflection of a line y = x is H_λ - reflection with $\lambda = \frac{1}{2}$.

The same argument is true for the line y = -x.

As a result, we have the set of reflections

$$S = \{(x, y) \mid x = 0, y = 0, y = x, \text{ and } y = -x\}.$$

In matrix form, we can write the reflections as

$$S = \left\{ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}$$

3.3. ROTATIONS

Let us, now, search for the rotations. We claim that there are only four rotations that preserves the taxicab distance in taxicab geometry.

Theorem 8. The isometric rotations in R^2_T are consists of

$$R_\theta = \left\{ A_\theta \mid \theta = k \frac{\pi_T}{2}, \; k = 0, 1, 2, 3 \right\}.$$

Proof. Let $P = (1, 0)$ and $Q = (0, 1)$.

Rotating P with an angle θ, we get $A_\theta(P) = (\cos_T\theta, \sin_T\theta)$ on the unit circle. If we also rotate Q with the angle θ, we get $A_\theta(Q) = (-\sin_T\theta, \cos_T\theta)$ (Figure 3). Since we want to rotation that preserves the distance and since $d_T(P, Q) = 2$, we need to have $d_T(A_\theta(P), A_\theta(Q)) = 2$.

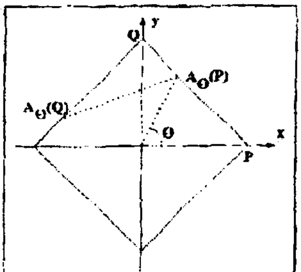

Figure 3

Thus

$$|\cos_T\theta + \sin_T\theta| + |\sin_T\theta + \cos_T\theta| = 1 + |\sin_T\theta - \cos_T\theta| = 2$$

which implies

$$|\sin_T\theta - \cos_T\theta| = 1.$$

It follows that

$$\sin_T\theta - \cos_T\theta = 1 \quad \text{or} \quad \sin_T\theta - \cos_T\theta = -1.$$

Thus,

$$\cos_T\theta = 0 \quad \text{or} \quad \sin_T\theta = 0 .$$

That means $\theta = \left\{ 0, \frac{\pi_T}{2}, \pi, \frac{3\pi_T}{2} \right\}$, as claimed.

Not that this argument is true for arbitrary P and Q.

We can also write the isometric rotations in matrix form as

$$R_\theta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

Thus, we have the orthogonal group, consisting of four reflections and four rotations,

$$O_T(2) = R_\theta \cup S.$$

Now, we would like to prove that all isometries of taxicab plane are $T(2)$, $O_T(2)$, and no others.

Theorem 9 (Main Theorem). Let $F : R^2_T \to R^2_T$ be an isometry. Then, there exists a unique $T_a \in T(2)$ and $C \in O_T(2)$ such that $F = T_a \circ C$.

To prove this theorem which is, in fact, the aim of this paper, we give the following definitions and theorems:

Definition 4. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$ be two points in R^2_T. The line segment from A to B, denoted by [AB], is defined by

$$[AB] = \{P \mid P = (a_1, a_2) + t(b_1 - a_1, b_2 - a_2)\ ,\ 0 \le t \le 1\}.$$

Definition 5. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$ the two points in R^2_T. The standard rectangle with diagonal [AB], denoted by $\overset{\square}{AB}$, is defined by

$$\overset{\square}{AB} = \{P \mid d_T(A,P) + d_T(P, B) = d_T(A,B)\}.$$

Corollary 10. Let $A = (a_1, a_2)$, $B = (b_1, b_2) \in R^2_T$. Then,

$$\overset{\square}{AB} = [a_1, b_1] \times [a_2, b_2].$$

Note: If the line segment [AB] is horizontal or vertical, then we define

$$\overset{\square}{AB} = [AB]\ .$$

Theorem 11. Let $F : R^2_T \to R^2_T$ be an isometry and let $\overset{\square}{AB}$ be the standard rectangle. Then,

$$F(\overset{\square}{AB}) = \overset{\square}{F(A)F(B)}\ .$$

Proof. Let $P \in F(\overset{\square}{AB})$. Then,

$$P \in F(\overset{\square}{AB}) \Leftrightarrow \exists\, C \in \overset{\square}{AB} \ni P = F(C)$$

$$\overset{\text{def } 4}{\Leftrightarrow} d_T(A, C) + d_T(C, B) = d_T(A, B)$$

$$\overset{\text{F iso.}}{\Leftrightarrow} d_T(F(A), F(C)) + d_T(F(C), F(B)) = d_T(F(A), F(B))$$

$$\overset{\text{def } 4}{\Leftrightarrow} P = F(C) \in \overset{\square}{F(A)F(B)}\ .$$

Corollary 12. Let $F : R^2_T \to R^2_T$ be an isometry and let $\overset{\square}{AB}$ be the standard rectangle. Then, F preserves to be a corner point and preserves the circumference of $\overset{\square}{AB}$.

Theorem 13. Let $f : R^2_T \to R^2_T$ be an isometry such that $f(0) = 0$. Then, $f \in R_\theta$ or $f \in S$.

Proof. Let $A = (1, 0)$, $B = (0, 1)$ and consider the standard rectangle $\overset{\square}{OD}$.

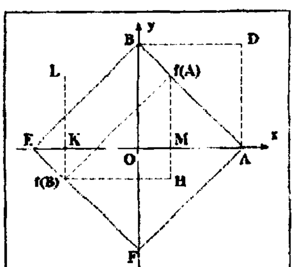

Figure 4

It is clear form the Figure 4 that

$$f(A) \in [AB] \quad \text{or} \quad f(A) \in [BE] \quad \text{or} \quad f(A) \in [EF] \quad \text{or} \quad f(A) \in [AF]\ .$$

CASE 1. Suppose $f(A) \in [AB]$ and suppose $f(A) \neq A$ and $f(A) \neq B$.

Since

$$d_T(A, B) = 2 \quad \text{and} \quad d_T(O, B) = 1,$$

it follows that

$$d_T(f(A), f(B)) = 2 \quad \text{and} \quad d_T(f(O), f(B)) = 1.$$

Therefore $f(B) \in [EF]$. Also,

$$1 = d_T(O, , f(B)) = d_T(f(B), K) + d_T(K, O)$$

which implies that $d_T(K, O) < 1$.

On the other hand, since D is a corner of $\overset{\square}{AB}$, by Theorem 6 and Corollary 7, f(D) is a corner of $f(A)\overset{\square}{}f(B)$. Also, since $d_T(K, O) < 1$ and $d_T(O,D) = 2$, it follows that $f(D) \neq H$ and thus $f(D) = L$. (The other corner is in ABEF. Why?). Finally, since $d_T(K, O) < 1$ and

$$d_T(L, K) = d_T(K, O) = d_T(O, L) = d_T(O, D) = 2$$

it follows that $d_T(L, K) > 1$. Notice also that

$$1 \geq d_T(M, f(A)) = d_T(L, K)$$

which is a contradiction. This implies that

$$f(A) = A \quad \text{or} \quad f(A) = B.$$

If $f(A) = A$, then f is an identity function which is a rotation with $\theta = 0$ or a reflection by x-axes. Thus, $f \in R_\theta$ or $f \in S$.

Suppose, now, $f(A) = B$. Then,

$$f(B) = A \quad \text{or} \quad f(B) = E \quad \text{or} \quad f(B) = F .$$

Subcase 1. If $f(B) = A$, then f is a reflection by the line $y = x$. Thus, $f \in S$.

Subcase 2. If $f(B) = E$, then f is a rotation of $\theta = \frac{\pi_T}{2}$. Thus, $f \in R_\theta$.

Subcase 3. We shall prove that $f(B) \neq F$.

Proof. Suppose $f(B) = F$. Since

$$f(\overset{\square}{AB}) = f(A)\overset{\square}{}f(B) = [BF]$$

it follows that $f(D) \in [BF]$ which implies $d_T(O, f(D)) \leq 1$. But, we have

$$2 = d_T(O, D) = d_T(O, f(D))$$

which is a contradiction. Therefore, $f(B) \neq F$.

Finally, we note that there are three more cases:

$f(A) \in [BE]$ or $f(A) \in [EF]$ or $f(A) \in [AF]$

which will give us the rest of the cases of R_θ, and S.

We again note that this is true for arbitrary A and B. Finally, let us prove the main theorem, Theorem 5.

Proof of Theorem 5.

Let $F(0) = a$. Define $C = T_{-a} \circ F$. Obviously, C is an isometry and $C(0) = 0$. Then, it follows from Theorem 8 that $C \in O_T(2)$ and thus,

$F = T_a \circ C$.

Uniqueness is trivial.

REFERENCES

[1] EKİCİ, C., KOCAYUSUFOĞLU, İ., AKÇA, Z., The Norm in Taxicab Geometry, OGU Math. Preprint 97.13..

[2] KOCAYUSUFOĞLU, İ., EKİCİ, C., AKÇA, Z., The Inner-Product in Taxicab Geometri, Mathematics, Mechanics and Informatics (Kazakh State National University), (1998).

[3] KRAUSE, E.F., Taxicab Geometry, Addison-Wesley, Menlo Park, NJ, (1975).

[4] RUDIN, W., Functional Analysis, Tata Mc Graw-Hill Pub. Comp. Ltd., New Delhi (1989).

[5] SOWELL, K.O., Taxicab Geometry-A New Slant, Mathematics Magazine, 62 (1989) 238-248.