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THE PROJECTION AREA IN THE LINES SPACE

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ABSTRACT

In this paper, the area vectors of the closed spherical indicator of a unit dual sphere and the parallel projection areas of the planar regions surrounded in the lines space of these the closed spherical indicators are calculated in terms of the components of G. Koenigs screw. In addition, a orthonormal trihedron fixed to the striction point X of an initial ruled surface and some relations were obtained for the parallel projection areas of the spherical indicator of this trihedron.

1. INTRODUCTION

After the work of Steiner [12] and H. Holditch [6] the first study about spherical motions was given by E.B. Elliott [2,3]. Blaschke defined the Steiner point and the Steiner vector for one-parameter closed spherical motions and gave an area formula equivalent to that of Steiner for the case of a sphere [1]. The parallel projection area of a closed spatial curve formed under the motion $B(c_1)$ defined along the closed spherical curve c_1 was given in [11]. The oriented lines in $\mathbb{R}^3$ are one-to-one correspondence with the points of the dual unit sphere $\mathbb{D}^3$ (E. Study). A dual point of $\mathbb{D}^3$ corresponds to a line in $\mathbb{R}^3$; two different points of $\mathbb{D}^3$ represent two skew-lines in $\mathbb{R}^3$. A differentiable curve on the unit dual sphere represents a ruled surface in $\mathbb{R}^3$. In section II we give the basic concepts of this method.

II. BASIC CONCEPTS

If a and a* are real number and $\varepsilon^2 = 0$, the combination $A = a+\varepsilon a^*$ is called a dual number. An oriented line in $\mathbb{R}^3$ may be given by two points on it, $\vec{x}$ and $\vec{y}$. If λ is any nonzero constant, the parametric equation of the line can be given in the form $\vec{y} = \vec{x} + \lambda\vec{a}$, where $\vec{a}$ is the direction vector of the line. If $\vec{a}^*$ denotes the moment of the vector $\vec{a}$ with respect to the origin O we have $\vec{a}^* = \vec{x} \wedge \vec{a} = \vec{y} \wedge \vec{a}$, where $\wedge$ denotes the exterior product of the vectors. This means that the direction vector $\vec{a}$ of the line and its moment vector $\vec{a}^*$ are independent of one another; they satisfy the following equations:

$$\langle \vec{a},\vec{a} \rangle = 1 \text{ and } \langle \vec{a},\vec{a}^* \rangle = 0 \tag{II.1}$$

where $\langle\ ,\ \rangle$ denotes the scalar points of the vectors. The six components a_i and a_i^* (i = 1,2,3) of $\vec{a}$ and $\vec{a}^*$ are Plückerian homogeneous line coordinates. Hence the vectors $\vec{a}$ and $\vec{a}^*$ determined the oriented line. The set of oriented lines in $\mathbb{R}^3$ is one-to-one correspondence with pairs of vectors in $\mathbb{R}^3$ subject to the condition (II.1), and so we may expect to represent it as a certain four dimensional set in $\mathbb{R}^6$ of six tuples of real numbers; we may take the space $\mathbb{D}^3$ of triples of dual numbers with coordinates $X_1 = x_1 + \varepsilon x_1^*$, $X_2 = x_2 + \varepsilon x_2^*$, $X_3 = x_3 + \varepsilon x_3^*$, $\varepsilon^2 = 0$. Each line in $\mathbb{R}^3$ is represented by the dual vector $\vec{A} = \vec{a} + \varepsilon\vec{a}^*$, $\langle \vec{A},\vec{A} \rangle = \langle \vec{a},\vec{a} \rangle + 2\varepsilon\langle \vec{a},\vec{a}^* \rangle = 1$ in $\mathbb{D}^3$. The norm of a dual vector $\vec{A} = \vec{a} + \varepsilon\vec{a}^*$ is the dual number $\|\vec{A}\| = \left(\langle \vec{A},\vec{A} \rangle\right)^{\frac{1}{2}} = \left(\|\vec{a}\|\ ,\ \varepsilon\ \frac{\langle \vec{a},\vec{a}^* \rangle}{\|\vec{a}\|}\right)$, $\vec{a} \neq \vec{0}$.

Thus the following Theorem of E. Study can be given [4]:

"The oriented lines in $\mathbb{R}^3$ are in one-to-one correspondence with the points of the dual unit sphere $\langle \vec{A},\vec{A} \rangle = 1$ in $\mathbb{D}^3$."

Let X be a point on the line $a = (\vec{a},\vec{a}^*)$ and $\vec{x}$ be its position vector with respect to a fixed point O. Then,

$$\vec{a}^* = \vec{x} \wedge \vec{a} \tag{II.2}$$

where $\wedge$ denotes the exterior product of the vectors. If the point X is also on the line $b = (\vec{b},\vec{b}^*)$ then

$$\langle \vec{a},\vec{b}^* \rangle + \langle \vec{a}^*,\vec{b} \rangle = 0$$

This is called the Klein Form or the reciprocal product. This gives us a necessary condition for intersection of the lines $(\vec{a},\vec{a}^*)$ and $(\vec{b},\vec{b}^*)$. In the Euclidean space the set of all the line $(\vec{a},\vec{a}^*)$ which satisfies the following equation

$$\langle \vec{a},\vec{q}^* \rangle + \langle \vec{a}^*,\vec{q} \rangle = 0$$

is called linear line complex. The dual unit vector $\vec{P} = \dfrac{\vec{Q}}{\|\vec{Q}\|} = \vec{p} + \varepsilon\vec{p}^*$ of the dual vector $\vec{Q} = \vec{q} + \varepsilon\vec{q}^*$ is called the axis of the linear line complex. After this the P and Q we will be denoted such that $P = (\vec{p},\vec{p}^*)$ and $Q = (\vec{q},\vec{q}^*)$, respectively. Where

$$\vec{p} = \frac{\vec{q}}{\|\vec{q}\|} \quad , \quad \vec{p}^* = \frac{\vec{q}^* - k\vec{q}}{\|\vec{q}\|} \quad , \quad k = \frac{\langle \vec{q},\vec{q}^* \rangle}{\langle \vec{q},\vec{q} \rangle} \tag{II.3}$$

and k is the pitch of the linear line complex [8].

The pair $(\vec{s},\vec{s}^*)$, which are formed by the integral vectors

$$\vec{s} = \oint \vec{q}\, dt \quad \text{and} \quad \vec{s}^* = \oint \vec{q}^*\, dt \tag{II.4}$$

is called G. Koenigs screw [8].

A surface which is generated by the motion of a straight line is called a ruled surface. The infinitude of straight lines which thus lie on the surface are called its "generators". Simple examples of ruled surfaces are cones, cylinders, hyperboloids of one sheet, surfaces formed by the tangents, principal normals or binormals of a curve in space. If the ruled surface

$$\varphi: I \times R \to E^3$$
$$(t,\lambda) \to \varphi(t,\lambda) = \vec{r}(t) + \lambda\vec{a}(t)$$

satisfies the condition

$$\varphi\ (t + 2\pi,\lambda) = \varphi(t,\lambda)\ , \text{ for } \forall t\in I$$

then the ruled surface is called closed ruled surface, where I is an interval in $\mathbb{R}$.

III. THE PROJECTION AREA IN THE LINES SPACE

Let R and R^1 be two spaces in the 3-dimensional Euclidean space. Let $B = R/R^1$ denote the motion of R with respect to R^1, where R is moving space and R^1 is fixed space. We may represent these spaces with the frames $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\{O^1; \vec{e}^{\,1}_1, \vec{e}^{\,1}_2, \vec{e}^{\,1}_3\}$, respectively. At the initial $O = O^1$ and $\vec{e}_i = \vec{e}^{\,1}_i$, $i = 1,2,3$.

Let $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the frame then it is a moving frame of the moving space R and this frame be of enough order differentiable with respect to t, $0 \le t \le T$. During the space motion $B = R/R^1$ the point $X \in R$ draws a trajectory with absolute velocity $\dot{\vec{x}}_a$ in a fixed space R^1. According to the velocity law

$$\dot{\vec{x}}_a = \dot{\vec{x}}_f + \dot{\vec{x}}_r \tag{III.1}$$

where $\dot{\vec{x}}_f$ is the sliding velocity of the point X and $\dot{\vec{x}}_r$ is also the relative velocity of X. If X is a fixed point in moving space R then

$$\dot{\vec{x}}_a = \dot{\vec{x}}_f = \dot{\vec{x}} \tag{III.2}$$

If we consider a helical motion around the axis P with the parameter k, this helical motion can be formed by a rolling around the axis P with Darboux rolling vector $\vec{q}$ and a sliding along the axis which has the sliding vector $k\vec{q}$. Here $\|\vec{q}\|$ is the rotation angle around the Darboux vector $\vec{q}$ and $k = \dfrac{\|k\vec{q}\|}{\|\vec{q}\|}$.

Let M be a point on the axis P such that $OM = \vec{m}$ (see Fig. 1).

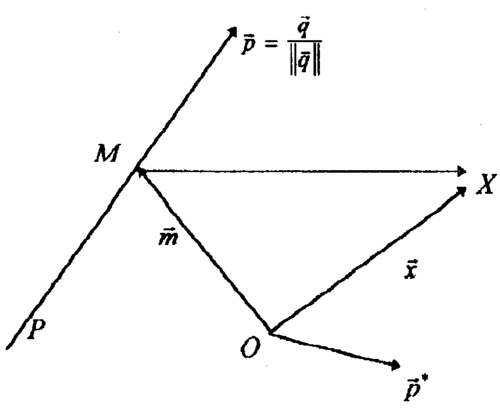

Fig. 1

$\vec{p}^* = \vec{m} \wedge \frac{\vec{q}}{\|\vec{q}\|}$ and from formula (II.3) we obtain

$$\vec{m} \wedge \vec{q} = \vec{q}^* - k\vec{q}$$

Let X be a point that makes helical motion around the axis P. The trajectory of this point is an ordinary helix. The velocity vector $\dot{\vec{x}}$ of the point X can be formed by only the velocity component of the sliding which occurs along the axis P; i.e.,

$$\dot{\vec{x}}_f = \dot{\vec{x}} = \dot{\vec{x}}_d + \dot{\vec{x}}_k$$

where $\dot{\vec{x}}_d$ is rotation velocity and $\dot{\vec{x}}_k$ is sliding velocity. Rotation velocity is

$$\dot{\vec{x}}_d = \vec{q} \wedge (\vec{x} - \vec{m})$$

since $\overrightarrow{MX} = \overrightarrow{OX} - \overrightarrow{OM} = \vec{x} - \vec{m}$. It is obvious that $\dot{\vec{x}}_k = k\vec{q}$. Therefore, the velocity of the point $X \in R$ during the motion $B = R/R^1$ can be obtained as

$$d^1\vec{x} = \dot{\vec{x}}_f = \dot{\vec{x}} = \vec{q}^* + \vec{q} \wedge \vec{x} \tag{III.3}$$

Let the straight line $a = (\vec{a},\vec{a}^*)$ be any normal of the helical trajectory of the point X. Then we get

$$\langle \vec{a},\vec{q}^* \rangle + \langle \vec{a}^*,\vec{q} \rangle = 0$$

since $\vec{a}^* = \vec{x} \wedge \vec{a}$ and $\langle \vec{a}^*,\dot{\vec{x}}_f \rangle = 0$. Now we can give the following theorem.

Theorem III.1. A helical motion is connected to a linear line complex and vice-versa. The axis of the complex is coincident with the axis of the helix. The parameter of the complex and the parameter of the helix are same. ∞^3 lines of a linear complex are formed by all normals of the trajectory at points which are suitable for helical motion of $\mathbb{R}^3$ [9].

Let us consider a fixed line $a = (\vec{a},\vec{a}^*)$ in the moving space R. During the closed spatial motion $B = R/R^1$ the velocity components of this line are

$$\dot{\vec{a}}_a = \dot{\vec{a}}_f = \dot{\vec{a}} \quad \text{and} \quad \dot{\vec{a}}_a^* = \dot{\vec{a}}_f^* = \dot{\vec{a}}^* \tag{III.4}$$

Let $\vec{q}$ be a Darboux rotation vector then

$$\dot{\vec{a}} = \vec{q} \wedge \vec{a} \quad \text{and} \quad \dot{\vec{a}}^* = \vec{q}^* \wedge \vec{a} + \vec{q} \wedge \vec{a}^* \tag{III.5}$$

The line a = $(\vec{a},\vec{a}^*)$ in the moving space R draws a ruled surface **A** during the closed spatial motion B = R/R^1. At the same time, the unit vector $\vec{a}$ draws spherical indicator c(A). Let X be a point in the moving space R and curve c(X) be the curve drawn by X, during the closed spatial motion B = R/R^1. The area vector of the curve c(X) in the fixed space R^1 is

$$\vec{V}_x = \oint \vec{x} \wedge \dot{\vec{x}}\, dt \tag{III.6}$$

[10], where $\vec{x}$ is the position vector of X and the integration is taken along the closed curve c(X) on R^1. Thus the following Theorem can be given [9]:

Theorem III.2. Let c(X) be the trajectory of a fixed point X of the moving space $\mathbb{R}$ in the fixed space $\mathbb{R}^1$. The projection area of the planar region produced by taking orthogonal projection onto a plane in the direction of the unit vector $\vec{n}$ of c(X) is

$$F_{X^n} = \langle \vec{n}, \vec{V}_X \rangle . \tag{III.7}$$

On the other hand, if F_{X^n} is the projection area of the planar region produced by taking orthogonal projection onto a plane and F_{X^P} is the projection area of the planar region happened by projecting onto a plane in any direction, then

$$F_{X^n} = \cos\varphi\, F_{X^P}$$

where φ is the angle between two image planes. Hence the area vector of the spherical indicator c(A) can be obtained as

$$\vec{V}_A = \vec{s} - \vec{a}\langle \vec{a}, \vec{s} \rangle \tag{III.8}$$

Let us consider $\vec{a}$, $\vec{b}$ and $\vec{c}$ trihedron which is fixed to a point X on the striction curve. Here the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ of a curve are dominator, central normal and central tangent vectors; respectively. Let **A** be an initial ruled surface which is formed by the line a = $(\vec{a},\vec{a}^*)$ during the motion B = R/R^1. The trihedron $\vec{a}$, $\vec{b}$ and $\vec{c}$ on the striction point X of the ruled surface **A** can be defined as follow [8]

$$\vec{b} = \frac{\dot{\vec{a}}}{\|\dot{\vec{a}}\|} = \frac{\vec{q} \wedge \vec{a}}{\sqrt{\langle \vec{q},\vec{q}\rangle - (\langle \vec{a},\vec{q}\rangle)^2}}$$

$$\vec{c} = \vec{a} \wedge \vec{b} = \frac{\vec{q} - \vec{a}\langle \vec{a},\vec{q}\rangle}{\sqrt{\langle \vec{q},\vec{q}\rangle - (\langle \vec{a},\vec{q}\rangle)^2}} \qquad \text{(III.9)}$$

where $\vec{a}$ is the dominator of the ruled surface **A**.

During the closed spatial motion $B = R/R^1$ the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form the initial ruled surface **A**, the central ruled surface **B** and the central tangent ruled surface **C**, respectively. During the same motion, the spherical indicators corresponding to these trihedron are c(A), c(B) and c(C).

Theorem III.3. Let a trihedron be $\vec{a}$, $\vec{b}$ and $\vec{c}$ which is fixed to striction point X of the initial ruled surface A. During the closed spatial motion $B = R/R^1$, the area vectors of the spherical indicator which corresponds to this trihedron are

$$\begin{aligned} \vec{V}_A &= \vec{s} - \vec{a}\langle \vec{a},\vec{s}\rangle \\ \vec{V}_B &= \vec{s} \\ \vec{V}_C &= \vec{a}\langle \vec{a},\vec{s}\rangle \end{aligned} \qquad \text{(III.10)}$$

respectively, where $\vec{s}$ is the Steiner vector of the motion $B = R/R^1$ [7].

Now we can give the following corollary:

Corollary III.4. Let c(A), c(B) and c(C) be the spherical indicator of the trihedron $\vec{a}$, $\vec{b}$ and $\vec{c}$ during the closed spatial motion $B=R/R^1$. If the area vectors of these spherical indicator denoted $\vec{V}_A$, $\vec{V}_B$ and $\vec{V}_C$, respectively, then the Steiner vector $\vec{s}$ of the motion $B = R/R^1$ can be obtained as

$$\vec{s} = \frac{1}{2}\left(\vec{V}_A + \vec{V}_B + \vec{V}_C\right)$$

since $\langle \vec{x},\vec{V}_X\rangle = \vec{0}$ we can give the following theorem.

Theorem III.5. Let $\vec{a}$ be the dominator vector, $\vec{b}$ be the central normal vector and $\vec{c}$ be the central tangent vector on the striction curve. Let c(A), c(B) and c(C) be their spherical indicator formed during the closed spatial motion $B = R/R^1$. If the area vectors of these spherical indicator are denoted $\vec{V}_A$, $\vec{V}_B$ and $\vec{V}_C$ then the projection of each of them on the central normal vector $\vec{b}$ is zero.

In addition, if the spherical indicators c(A), c(B) and c(C) is projected as parallel on a plane have the following corollary.

Corollary III.6. Let c(B) be the spherical indicator of the central normal vector $\vec{b}$. Then

$$F_{B^p} = F_{A^p} + F_{C^p} ,$$

where F_{A^p}, F_{B^p} and F_{C^p} are oriented parallel projection area of spherical indicator c(A), c(B) and c(C), respectively.

Let **A** be an initial ruled surface and $\mathfrak{I}$ be a torus drawn by the straight lines $a = (\vec{a},\vec{a}^*)$ and $g = (\vec{g},\vec{g}^*)$, respectively. Moreover surrounded the ruled surface **A** by the torus $\mathfrak{I}$.

Let φ be the angle between the generator of the ruled surface **A** and the dominator vector of the torus $\mathfrak{I}$. Hence, the vector $\vec{g}$ can be denoted by

$$\vec{g} = \sin\varphi\,\vec{b} + \cos\varphi\,\vec{c} \quad , \quad \varphi = \text{constant}$$

on the plane (b,c). During the closed spatial motion $B = R/R^1$, the spherical indicator c(G) is drawn on the sphere as unit vector $\vec{g}$ forms the torus $\mathfrak{I}$ which surrounds the ruled surface **A**. The area vector of the spherical indicator c(G) is

$$\vec{V}_G = \sin^2\vec{V}_B + \cos^2\vec{V}_C$$

Thus, we can give the following theorems:

Theorem III.7. Let us consider the planar region, which is formed by the parallel projection of spherical indicator c(G), on a plane. Its oriented projection area F_{G^p} is

$$F_{G^p} = \sin^2\varphi\, F_{B^p} + \cos^2\varphi\, F_{C^p}$$

Theorem III.8. The projection of the area vector $\vec{V}_G$ of the spherical indicator c(G) on the central normal vector $\vec{b}$ is zero.

Corollary III.9. If c(B) = c(C) then $F_{G^p} = F_{B^p}$.

Now let us consider the unit vector $\vec{h}$ on the plane (a,b). $\vec{h}$ can be denoted

$$\vec{h} = \sin\theta\, \vec{a} + \cos\theta\, \vec{b} \quad , \quad \theta = \text{constant}$$

In the line space, during the one-parameter closed spatial motion $B = R/R^1$, while vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{h}$ draws the ruled surfaces **A**, **B**, **C** and **H**, respectively, the end point of each of these vectors also draws the spherical indicator c(A), c(B), c(C) and c(H) on the sphere, respectively. Then the area vector of the spherical indicator c(H) can be obtained as

$$\vec{V}_H = \sin^2\theta \vec{V}_A + \cos^2\theta \vec{V}_B$$

Consider a unit vector $\vec{w}$ on the plane (a,c), such that $\vec{w}$ generates a torus Ω which surrounds the central normal ruled surface **B**. The vector $\vec{w}$ can be given by

$$\vec{w} = \cos\alpha\, \vec{a} + \sin\alpha\, \vec{c} \quad , \quad \alpha = \text{constant}$$

In the line space, during the one-parameter closed spatial motion $B = R/R^1$, while vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{w}$ draws the ruled surfaces **A**, **B**, **C** and Ω, respectively, the end point of each of these vectors also draws the spherical indicator c(A), c(B), c(C) and c(W) on the sphere, respectively. Then, the area vector of the spherical indicator c(W) can be obtained as

$$\vec{V}_W = \cos^2\alpha \vec{V}_A + \sin^2\alpha \vec{V}_C$$

Hence we can give the following corollary.

Corollary III.10. Let us consider the planar region, which is formed by the parallel projection of the spherical indicator c(H), on a plane. Its oriented projection area F_{H^p} is

$$F_{H^p} = \sin^2\theta F_{A^p} + \cos^2\theta F_{B^p}$$

Corollary III.11. Let us consider the planar region, which is formed by the parallel projection of the spherical indicator c(W), on a plane. Its oriented projection area F_{W^p} is

$$F_{W^p} = \cos^2 aF_{A^p} + \sin^2 aF_{C^p}$$

Corollary III.12. If $\vec{V}_H$ and $\vec{V}_W$ are the area vectors of the spherical indicators c(H) and c(W) then $\langle\vec{b},\vec{V}_H\rangle = 0$ and $\langle\vec{b},\vec{V}_W\rangle = 0$.

Corollary III.13. c(B) = c(C) then $F_{W^p} = F_{B^p}$.

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