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k - CONVEXITY IN EUCLIDEAN 3 - SPACE

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ABSTRACT

In this paper we establish some results relating the concept of k–convexity of regions in the Euclidean 3–space E^3 with the sectional curvature of their boundaires. Relation between k–convexity and strict convexity is considered. Focal points of the boundary of a k–convex region are also studied.

INTRODUCTION

Convexity in the Euclidean n–space E^n $(n \geq 2)$ is defined as follows: A subset $A \subset E^n$ is convex if for each pair of points $p, q \in A$ the closed segment $[pq]$ joining p and q is contained wholly in A. Other types of convexity such as strong, strict as well as weak convexity in general Riemannian manifolds are given in [1].

From now on a region $\Omega \subset E^n$ will be taken as an open subset of E^n with the property that any two points of Ω may be joined by a curve in Ω, i.e.a region is an open arcwise connected subset of E^n.

The concept of a k–convex region Ω with boundary $\partial\Omega$ in E^2 is defined for the first time in a remarkable paper [4] by D. Mejia and D. Minda in the following way:

Suppose $k \in [0, \infty)$. A region $\Omega \subset E^2$ is called k–convex provided $d(a, b) < \frac{2}{k}$ for any pair of points $a, b \in \Omega$ (where d denotes the distance function) and $E^2_k[a, b] \subset \Omega$. $E^2_k[a, b]$ is the intersection two closed disks of radius $\frac{1}{k}$ having a, b on their boundaries (see [4]).

The above mentioned concept of convexity has not yet been generalized for regions in Euclidean 3-space E^3. Consequently, this paper is devoted to study such a case.

Let a, b be any two points in E^3 with distance $d(a, b) < \frac{2}{k}$ apart for some positive real k and $\bar{D}$ be a closed flat disk of radius $\frac{1}{k}$ such that a, b are on the circular boundary $\partial\bar{D}$. The closed line segment [ab] divides $\bar{D}$ into two areas one of them is smaller than the other. Rotate the smaller area about the straight line $\overleftrightarrow{ab}$ through a and b. Then the resulting closed volume from this rotation will be denoted by $E^3_k\,[a, b]$. We also take $E^3_0\,[a, b]$ to be the closed line segment [ab]. If we allow the distance between a and b to satisfy $d(a, b) = \frac{2}{k}$, $E^3_k\,[a, b]$ will be the closed geodesic ball $\overline{B}_{1/k}\left(\frac{a+b}{2}\right)$ with center $\frac{a+b}{2}$ and radius $\frac{1}{k}$. For $0 \leq k' < k \leq \frac{2}{a+b}$ one can can easily see that $E^3_{k'}\,[a, b] \subset E^3_k\,[a, b]$. We can also see that $\partial E^3_{k'}\,[a, b] \cap \partial E^3_k\,[a, b] = \{a, b\}$.

Definition 1.

Suppose that $k \in [0, \infty)$. A region Ω in E^3 is called k-convex if for each pair of points $a, b \in \Omega$

(i) $d(a, b) < \frac{2}{k}$ and (ii) $E^3_k\,[a, b] \subset \Omega$.

From this definition we can easily prove the following results:

(a) An open ball in E^3 of radius $\frac{1}{k} > 0$ is a k-convex region,

(b) o-convexity is exactly the convexity defined above,

(c) The intersection two k-convex regions is k-convex,

(d) For the two regions Ω_1 and Ω_2 (none of the them is included in the other) which are k_1-convex and k_2-convex, respectively, where $k_1 > k_2$ we have that $\Omega_1 \cap \Omega_2$ is a k_2-convex region in E^3,

(e) Every k-convex region $\Omega \subset E^3$ is k'-convex for $k' < k$. Hence, every k-convex region Ω ($k \geq 0$) is o-convex. In other words every k-convex region is convex.

For a smooth surface $M \subset E^3$ let us write $K(x, M)$ to denote the sectional (Gaussian) curvature of M at the point $x \in M$.

The following proposition gives sufficient conditions for k-convexity.

Proposition 2. Suppose that Ω is simply connected region in E^3 bounded by the smooth surface $\partial\Omega$ diffeomorphic to the unite sphere S^2 and $K(c, \partial\Omega) \geq k^2 > 0$ for every $c \in \partial\Omega$. Suppose that when the circle $C(p, \lambda)$ of center p and radius $\lambda > 0$ lies locally at $q \in C(p, \lambda)$ in $\bar{\Omega}$ implies that the sphere $S(p, \lambda)$ lies locally at q in $\bar{\Omega}$. Then Ω is k-convex.

Proof: The hypotheses imply that Ω is convex according to the following result proved by R. Sacksteder [5].

Let M be a complete, Riemannian n-manifold ($n \geq 2$) and let $x: M \to E^{n+1}$ be a C^{n+1} isometric immersion. Suppose that every sectional curvature of M is non-negative, and at least one is positive. Then the image x(M) is the boundary of a convex body in E^{n+1}.

Notice that the last condition of Sacksteder's result is satisfied as $\partial\Omega$ is a compact hypersurface of E^3.

First we show that if $\bar{B}$ is any closed geodesic ball that is contained in the closure $\bar{\Omega}$ of Ω, then the radius of $\bar{B}$ is at most $1/k$.

Suppose $\bar{B} = \{x \in E^3: d(x, a) \leq r\}$ where $a \in \Omega$ and $\delta = \delta\Omega(a)$ the distance from a to $\partial\Omega$. Since $\bar{E} \subset \Omega$, then $r \leq \delta$ and so it sufficies to show that $\delta \leq \dfrac{1}{k}$.

Suppose that $c \in \partial\Omega \cap \{z: d(z, a) = \delta\}$. The geodesic sphere $S_\delta(a) = \{z \in \bar{\Omega}: d(z, a) = \delta\}$ lies inside of or on $\partial\Omega$ and consequently $S_\delta(a)$ and $\partial\Omega$ intersect tangentially at c. According to a well known result of convexity (see [2]) the tangent hyperplane $T_c(\partial\Omega)$ of $\partial\Omega$ at c represents a global support of $\bar{\Omega}$ (see Fig. (1)). Utilizing the height

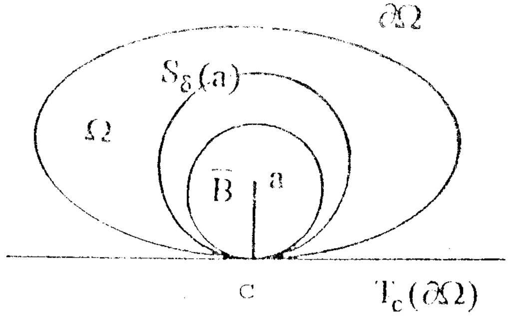

Fig. (1)

function concept of both $\partial\Omega$ and $S_\delta(a)$ at c with respect to T_c $(\partial\Omega)$ (see [2]) we can prove that $k^3 \leq K(c, \partial\Omega) \leq K(c, S_\delta(a)) = \frac{1}{\delta^2}$

Consequently $\delta \leq \frac{1}{k}$.

Next we show that $E^3_{\tilde{k}}$ [a, b] is contained in $\overline{\Omega}$ for arbitrary points a, $b \in \Omega$.

Since Ω is convex, then $[ab] \subset \Omega$. Because Ω is open there exists a real number $k' > 0$ such that $E^3_{k'}$ $[a, b] \subset \Omega$. Let $\tilde{k} = \sup \{k'$: $E^3_{k'}$, $[a, b] \subset \Omega\}$. Note that $E^3_{\tilde{k}}$ [a, b] is contained in the closure of Ω. Since $k' < \frac{2}{d(a, b)}$ for all admissible k', we must have $\tilde{k} \leq \frac{2}{d(a, b)}$. Now we discuss these two possibilities separately.

(i) If $\tilde{k} = \frac{2}{d(a, b)}$, then the closed geodesic ball $\overline{B}_{1/\tilde{k}}\left(\frac{a+b}{2}\right)$ centered at $\frac{a+b}{2}$ with radius $\frac{1}{\tilde{k}}$ lies in $\overline{\Omega}$. From the first part of the proof we obtain $\frac{1}{\tilde{k}^2} \leq \frac{1}{k^2}$, or $k \leq \tilde{k}$. In this way and as mentioned before E^3_k [a, b] $\subset E^3_{\tilde{k}}$ [a, b] which imlies that E^3_k [a, b] $\subset \bar{\Omega}$.

(ii) The other possibility is $\tilde{k} < \frac{2}{d(a, b)}$.

Let $c \in \partial\Omega \subset E^3_k$ [a, b], $c \neq a$, $c \neq b$. By hypothesis we can easily draw a sphere S of radius $\frac{1}{\tilde{k}}$ which containes $E^3_{\tilde{k}}$ [a, b] inside and intersects $\partial E^3_{\tilde{k}}$ [a, b] tangentially at c. Using the height function concept again at c for both S and $\partial\Omega$ we have

$$k^2 \leq K(c, \partial\Omega) \leq K(c, S) = \tilde{k}^2.$$

Hence $k \leq \tilde{k}$ and E^3_k [a, b] $\subset E^3_{\tilde{k}}$ [a, b].

To complete the proof it remains to show that E^3_k [a, b] $\subset \Omega$.

Let $\overleftrightarrow{ab}$ be the straight line through a and b as before. Select distinct points $a', b' \in (\Omega \cap \overleftrightarrow{ab}) \setminus [ab]$ such that $[ab] \subset [a'b']$. In the light of the above discussion E^3_k [a′, b′] $\subset \bar{\Omega}$. Since E^3_k [a, b] is contained in the interior of E^3_k [a′, b′], then E^3_k [a, b] $\subset \Omega$ and the proof is complete.

In the following we show that the converse of Proposition 2 is valid. We first need to prove the following result.

Lemma 3. (i) Supose that Ω is a k-convex region in E^3. Then for any $a \in \Omega$ and $c \in \partial\Omega$, E^1_k [a, c] $\setminus \{c\} \subset \Omega$.

(ii) If $a, c \in \partial\Omega$, then Int E^3_k [a, c] $\subset \Omega$.

Proof: Since Ω is k-convex then Ω is convex. It is not difficult to show that any geodesic ray emanating from $a \in \Omega$ intersecting $\partial\Omega$ will meet $\partial\Omega$ transversally at a single point. Consequently the half open segment [ac) will be contained in Ω. Note that $d(a, c) < \frac{2}{k}$ since $d(a, c) = \frac{2}{k}$ would imply that there exist $a' \in \Omega$ near a and $c' \in \Omega$ near c with

$d(a', c') \geq \frac{2}{k}$ which violates the definition of k–convexity.

As a second step we show that Int $E^3{}_k [a, c] \subset \Omega$. Take $c_n \in [ac)$ such that $c_n \to c$. Now $c_n \in \Omega$, so by k–convexity of Ω we have $E^3{}_k [a, c_n] \subset \Omega$ for all n. Hence Int $E^3{}_k [a, c] \subset \cup E^3{}_k [a, c_n] \subset \Omega$.

Select $a' \in \Omega$ such that $a \in (a'c)$. By the first part of the proof we get Int $E^3{}_k [a', c] \subset \Omega$. Because $E^3{}_k [a, c] \setminus \{c\} \subset$ Int $E^3{}_k [a', c]$ then $E^3{}_k [a, c] \setminus \{c\} . \subset \Omega$.

(ii) Let $a_n \in \Omega$ with $a_n \to a$. From (i) $E^3{}_k [a, c] \setminus \{c\} \subset \Omega$ for all n. Hence Int $E^3{}_k [a, c] \subset \Omega$.

Proposition 4. Suppose that Ω is a region in E^3 bounded by a smooth hypersurface $\partial\Omega$ diffeomorphic to S^2. If Ω is k–convex, then $K(c, \partial\Omega) \geq k^2 \geq 0$ for all $c \in \partial\Omega$.

Proof: Since Ω is k–convex, then for any two points $a, c \in \partial\Omega$ we have Int $E^3{}_k [a, c] \subset \Omega$ (Lemma (3)–(ii)). Let us consider all sufficiently small curves on $\partial\Omega$ starting from c and resulting from the intersection of $\partial\Omega$ with normal planes to $\partial\Omega$ through c. Let γ be one of these curves and $c_1, c_2, \ldots, c_n, \ldots \in \gamma$ a sequence of points such that $c_n \to c$. In the light of the above discussion we have Int $E^3{}_k [c_n, c] \subset \Omega$ for all n. Repeating this process with all the above mentioned curves we finally obtain a sphere S of radius $\frac{1}{k}$ tangent to $\partial\Omega$ at c which lies locally in E^3 / Ω as indicated in Fig. (2).

(We shall prove (proposition (6) that S lies globally in $E^3 \setminus \Omega$). Using the height function concept at c, we have that.

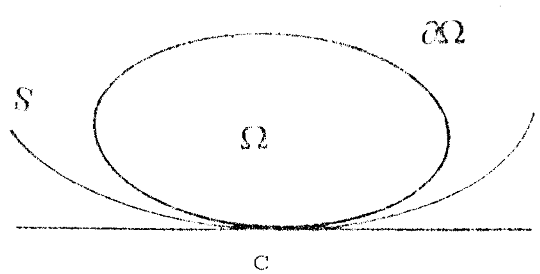

Fig. (2)

$K(c, \partial\Omega) \geq K(c, S) = k^2 \geq 0$ for all $c \in \Omega$ and the proof is complete.

Lemma 5. Suppose that D is an open geodesic ball of radius $\frac{1}{k}$, B is an open ball (or half-space of E^3) such that $c \in \partial B \cap \partial D$ and B and D are externally tangent at c. If $d(a, c). < \frac{2}{k}$ and $a \notin \bar{D}$, then $(E^3_k [a, c] \setminus \{c\}) \cap B \neq \varnothing$.

Proof: Let e be the center of the geodesic ball D and H the hyperplane determined by the triple a, c, e. The intersection of H with both D and B ($\bar{D}$ and $\bar{B}$, respectively) will lead to a situation in H similar to that given in lemma (0) [4]. Consequently, $(E^3_k [a, c] / \{c\}) \cap \bar{B} \neq \varnothing$. Rotating $E^2_k [a, c]$ about $[ac]$ we obtain $E^2_k [a, c]$ satisfying $(E^3_k [a, c] / \{c\}) \cap B = \varnothing$.

Proposition 6. Suppose that Ω is a k-convex region in E^3. Assume that $a \in \Omega$, $c \in \partial\Omega$ such that $d(a, c) = d(a, \Omega) = \delta_\Omega(a) = \delta$. If D is the open ball of radius $\frac{1}{k}$ whose boundary ∂D is tangent to the sphere $S\delta(a)$ at c and that D contains a in its interior, then $\Omega \subset D$.

Proof. Let $H = T_c\partial\Omega$ be the hyperplane of E^3 which is tangent to $\partial\Omega$ at c and E the open half-space determined by H that does not contain a (See Fig. (3)).

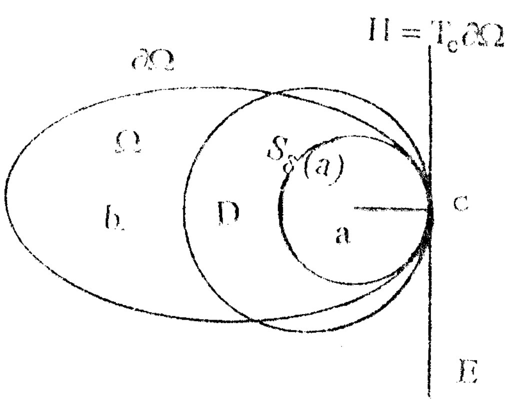

Fig. (3)

As Ω is convex then $\Omega \cap E = \varnothing$. Assume on contrary to the proposition that $\Omega \not\subset D$, i.e $\Omega \setminus D = \varnothing$. Then there exists a point $b \in \Omega \setminus D$. Because Ω is open, we may take $b \in \Omega \setminus \bar{D}$. Since Ω is k–convex, then $d(b, c) < \frac{2}{k}$. Using lemma (3) we have

$E^3_k [b, c] \setminus \{c\} \subset \Omega$ while lemma (5) gives that
$(E^3_k [b, c] \setminus \{c\}) \cap E \neq \varnothing$
contradicting $\Omega \cap E = \varnothing$. This contradiction shows that $\Omega \subset D$.

Strict convexity in Euclidean space may be defined as follows: An open subset $B \subset E^n$ $(n \geq 1)$ is strictly convex if

(i) B is convex.

(ii) For each pair of points $a, b \in \partial B$, $(ab) \subset B$.

In what follows we establish the relation between the k–convexity and strict convexity in E^3. It will become clear in the light of the following discussion that k–convexity $(k > 0)$ is stronger than strict convexity.

Proposition 7. Every k–convex region $(k > 0)$ in E^3 is a strictly convex subset.

Let Ω be a k–convex region E^3. Assume on contrary that Ω is not strictly convex. Consequently there exists a pair of points p, $q \in \partial\Omega$ such that $[pq] \subset \partial\Omega$. Let c be a point on the open segment (pq). It is an easy exercise to show that $K(c, \partial\Omega) = 0$ contradicting proposition (4).

The converse of the above proposition is not necessarily true according to the following example.

Let Ω be a region in E^3 with smooth boundary $\partial\Omega = S_1 \cup S_2$ where S_1 is a part of a sphere and S_2 is the regular path (See Fig. (4)) $x(u, v) = (u, v, (u^2 + v^2)^2)$, $u^2 + v^2 \leq 1$. Clearly Ω is a strictly convex subset of E^3 while $K(0, \partial\Omega) = 0$.

Proposition 8. Let Ω be a k–convex region $(k > 0)$ in E^3 and L the inner normal ray starting from $p \in \partial\Omega$. Take $p_0 \in L$ such that $d(p, p_0) = \frac{1}{k}$. Then there are two focal points of $\partial\Omega$ on $[pp_0]$.

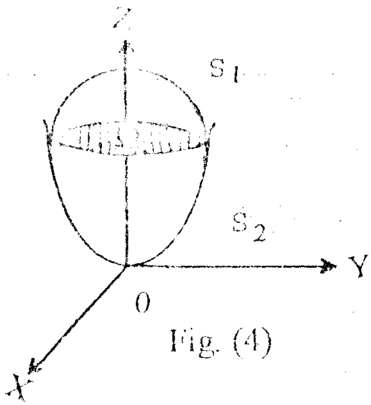

Fig. (4)

Proof. Since Ω is k-convex, $p \in \partial\Omega$, then by proposition (6) the open ball $B_{\frac{1}{k}}(p_0)$ centered at p_0 with radius $\frac{1}{k}$ contains Ω and

$p \in \partial\Omega \cap S_{\frac{1}{k}}(p_0.)$ The point p is a global strict maximum point of the distance function d(p',.) from each p' to $\partial\Omega$ where $p' \in L$, $d(p, p') > \frac{1}{k}$. Then L contains two focal points on the segment $[p_0 p]$. If the two focal point coincide, we have a focal point of $(p_0 p]$ of multiplicity 2, [3].

Notice that the condition $k > 0$ in the last Proposition 8 is essential as in the above example there are no focal points of $\partial\Omega$ on the z-axis.

Proposition 9. Let Ω be a k-convex region in E^3. Then every plane section of Ω is a k-convex region in E^3.

Proof. Let Π be a plane in E^3 such that $\Omega \cap \Pi \neq \varnothing$. Take $\Omega \cap \Pi = \Omega^*$. Let a, b be arbitrary points in Ω^*. Hence $d(a, b) < \frac{1}{k}$ as $a, b \in \Omega$.

Since Ω is k-convex, $E^3{}_k \subset \Omega$. Also $E^3{}_k [a, b] \cap \Pi = E^2 [a, b] \subset \Omega^*$ which completes the proof that Ω^* is a k-convex region in $E^2 = \Pi$.

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