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ON THE GEOMETRY OF TIME-LIKE SURFACES

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ABSTRACT

The formulae for a time-like surface, given by Enneper, O. Bonnet, Euler and Liouville, were obtained by using the hyperbolic angle between time-like curves on the surface.

1. INTRODUCTION

In the Euclidean 3-space E^3, let us denote the tangent unit vectors of the parameter curves (c_1), (c_2) and an arbitrary curve (c) passing through a point P on the differentiable surface x(u,v) as t_1, t_2 and t, respectively. Let φ be the angle between t_1 and t, R_1 and R_2 the principal curvature radii of (c_1) and (c_2). If R_n and T_g are the normal curvature and jeodesic torsion radii corresponding to direction t then we can write

$$\frac{\cos\varphi}{R_1} = \frac{\cos\varphi}{R_n} - \frac{\sin\varphi}{T_g}.$$

$$\frac{\sin\varphi}{R_2} = \frac{\sin\varphi}{R_n} + \frac{\cos\varphi}{T_g}. \qquad (1)$$

Now, Let us consider the Minkowski 3-space R_1^3 provided with Lorentzian inner product

$$\langle \mathbf{a},\mathbf{b}\rangle = a_1b_1 + a_2b_2 - a_3b_3$$

where $\mathbf{a} = (a_1,a_2,a_3)$, $\mathbf{b} = (b_1,b_2,b_3) \in R^3$. In this space, a vector **a** is said to be space-like if $\langle \mathbf{a},\mathbf{a}\rangle > 0$, time like if $\langle \mathbf{a},\mathbf{a}\rangle < 0$, and light-like (or null) if $\langle \mathbf{a},\mathbf{a}\rangle = 0$. The norm of a vector **a** is defined to be $|\mathbf{a}| = \sqrt{|\langle \mathbf{a},\mathbf{a}\rangle|}$ [2].

Let y(u,v) be a surface in the space R_1^3. If for each $P \in y(u,v)$ the induced metric $\langle\ ,\ \rangle|_{T_p y \times T_p y}$ is Lorentzian then y(u,v) is called time-like surface. The Darboux instantaneous rotation vector of space-like and time-like curves on the time-like surface y(u,v) were stated in [3]. By using this vectors, we give:

Fundamental theorem. Let us denote tangent unit vectors of time-like and space-like parameter curves (c_1) and (c_2) perpendicular to each other, and an arbitrary time-like curve (c) passing through a point P on the time-like surface y(u,v) as t_1, t_2 and t, respectively. Let θ be the hyperolic angle between t_1 and t, and also R_1, R_2 the principal curvature radii. If R_n and T_g are the normal curvature and geodesic torsion radii corresponding to direction t, then we can write

$$\frac{\cosh\theta}{R_1} = \frac{\cosh\theta}{R_n} - \frac{\sinh\theta}{T_g}\,,$$

$$\frac{\sinh\theta}{R_2} = \frac{\sinh\theta}{R_n} - \frac{\cosh\theta}{T_g}\,. \tag{1.1}$$

2. THE INSTANTANEOUS ROTATION VECTORS OF SPACE-LIKE AND TIME-LIKE CURVES

a) The Frenet instantaneous rotation vector

i) Let c = c(s) be a space-like space curve. At every point on this curve, there exist the Frenet trihedron [t,n,b], here t, n and b are tangent, principal normal and binormal unit vectors of curve, respectively. In this trihedron, we assume that t and n are space-like vectors and b is time-like vector. That is we have

$$-\langle t,t\rangle = \langle n,n\rangle = 1,\ \langle b,b\rangle = -1$$

$$\langle t,n\rangle = \langle n,b\rangle = \langle b,t\rangle = 0.$$

Furthermore, for this vectors we write

$$t \wedge n = b\ ,\ n \wedge b = -t\ ,\ b \wedge t = -n,$$

where $\wedge$ is the Lorentzian cross product [4] in space R_1^3. In this situation, the Frenet formulae are given by

$$\left.\begin{aligned} \frac{dt}{ds} &= \frac{1}{R}\,n \\ \frac{dn}{ds} &= -\frac{1}{R}\,t + \frac{1}{T}\,b \\ \frac{db}{ds} &= \frac{1}{T}\,n\,, \end{aligned}\right\} \tag{2.1}$$

where R and T are the radii of curvature and torsion of the space-like curve (c), respectively. By (2.1) the Frenet instantaneous rotation vector is given by

$$f = \frac{1}{T}\,t - \frac{1}{R}\,b\,. \tag{2.2}$$

ii) In the trihedron [t,n,b], let n be time-like vector. Thus, the Frenet formulae for the curve (c) are given by

$$\left.\begin{aligned} \frac{dt}{ds} &= \frac{1}{R}\,n \\ \frac{dn}{ds} &= \frac{1}{R}\,t + \frac{1}{T}\,b \\ \frac{db}{ds} &= \frac{1}{T}\,n\,. \end{aligned}\right\} \tag{2.3}$$

By (2.3) the Frenet instantaneous rotation vector for the space-like curve (c) can be written as follows:

$$f = -\frac{1}{T}\,t + \frac{1}{R}\,b\,. \tag{2.4}$$

The Frenet derivative formulae (2.3) can be given with the vector (2.4) as follows:

$$\left.\begin{aligned} \frac{dt}{ds} &= f \wedge t \\ \frac{dn}{ds} &= f \wedge n \\ \frac{db}{ds} &= f \wedge b\,. \end{aligned}\right\} \tag{2.5}$$

iii) Let (c) be a time-like curve.Then, the Frenet formulae are given by

$$\left.\begin{aligned} \frac{dt}{ds} &= \frac{1}{R}\,n \\ \frac{dn}{ds} &= \frac{1}{R}\,t - \frac{1}{T}\,b \\ \frac{db}{ds} &= \frac{1}{T}\,n\,. \end{aligned}\right\} \tag{2.6}$$

By (2.6), the Frenet instantaneous rotation vector of the time-like curve (c) can be written by

$$f = \frac{1}{T}\,t - \frac{1}{R}\,b\ . \tag{2.7}$$

b) The Darboux instantaneous rotation vector

i) let us consider the time-like surface y(u,v). At every point of a time-like curve (c) on this surface there exists a Frenet trihedron [t,n,b]. Since the curve (c) is on the surface, another trihedron can be mentioned. Let us denote the tangent unit vector of the curve (c) as t and the space-like normal unit vector of the surface at the ponit P as N. In this case, if we consider a space-like vector g, which is defined as $t \wedge N = g$, we obtain the Darboux trihedron [t,g,N]. To compare this trihedron with the Frenet trihedron, let us denote the angle between the vectors n and N as φ. In this situation, we can write

$$\begin{aligned} g &= n \sin\varphi - b \cos\varphi\,, \\ N &= n \cos\varphi + b \sin\varphi\ . \end{aligned} \tag{2.8}$$

Differentiating the vectors t, N and g, with respect to arc s of the curve (c) we obtain the formulae

$$\begin{aligned} \frac{dt}{ds} &= \rho \sin\varphi\, g + \rho \cos\varphi\, N \\ \frac{dg}{ds} &= \rho \sin\varphi\, t - \left(\tau - \frac{d\varphi}{ds}\right) N \\ \frac{dN}{ds} &= \rho \cos\varphi\, t + \left(\tau - \frac{d\varphi}{ds}\right) g\ . \end{aligned} \tag{2.9}$$

Here, if we say

$$\rho\cos\varphi = \frac{\cos\varphi}{R} = \frac{1}{R_n} = \rho_n$$

$$\rho\sin\varphi = \frac{\sin\varphi}{R} = \frac{1}{R_g} = \rho_g$$

$$\tau - \frac{d\varphi}{ds} = \frac{1}{T} - \frac{d\varphi}{ds} = \frac{1}{T_g} = \tau_g$$

the formulae (2.9) can be written as follows:

$$\frac{dt}{ds} = \rho_g g + \rho_n N$$
$$\frac{dg}{ds} = \rho_g t - \tau_g N \qquad (2.10)$$
$$\frac{dN}{ds} = \rho_n t + \tau_g g ,$$

where ρ_n is the normal curvature, ρ_g is the geodesic curvature and τ_g is the geodesic torsion.

For this, the Darboux instantaneous rotation vector of the Darboux trihedron can be written as below:

$$w = \frac{t}{T_g} + \frac{g}{R_n} - \frac{N}{R_g} . \qquad (2.11)$$

According to this, the Darboux derivative formulae are written as follows:

$$\frac{dt}{ds} = w \wedge t , \frac{dg}{ds} = w \wedge g , \frac{dN}{ds} = w \wedge N . \qquad (2.12)$$

ii) Let (c) be a space-like curve on the time-like surface. In trihedron [t,g,N], we assume that t and N are space-like vectors and g is time-like vector. Then the Lorentzian cross product for this vectors is given by

$$t \wedge g = -N , g \wedge N = -t , N \wedge t = g. \qquad (2.13)$$

Let θ be the hyperbolic angle [5] between the time-like unit vectors n and g. In this case, we have

$$N = n \sinh\theta + b \cosh\theta ,$$
$$g = n \cosh\theta + b \sinh\theta . \qquad (2.14)$$

Differentiating the t, g and N according to the arc s of curve (c) we obtain the following formulae:

$$\frac{dt}{ds} = \rho \cosh\theta \, g - \rho \sinh\theta \, N$$
$$\frac{dg}{ds} = \rho \cosh\theta \, t + \left(\tau + \frac{d\theta}{ds}\right) N \qquad (2.15)$$
$$\frac{dN}{ds} = \rho \sinh\theta \, t + \left(\tau + \frac{d\theta}{ds}\right) g .$$

Here, if we replace

$$\rho\cosh\theta = \frac{\cosh\theta}{R} = \frac{1}{R_g} = \rho_g$$

$$\rho\sinh\theta = \frac{\sinh\theta}{R} = \frac{1}{R_n} = \rho_n$$

$$\tau + \frac{d\theta}{ds} = \frac{1}{T} + \frac{d\theta}{ds} = \frac{1}{T_g} = \tau_g$$

then the Darboux derivative formulae are given by

$$\left.\begin{aligned} \frac{dt}{ds} &= \rho_g g - \rho_n N \\ \frac{dg}{ds} &= \rho_g t + \tau_g N \\ \frac{dN}{ds} &= \rho_n t + \tau_g g\ , \end{aligned}\right\} \tag{2.16}$$

where ρ_g is the geodesic curvature, ρ_n is the normal curvature and τ_g is the geodesic torsion.

Consequently, we can write the Darboux instantaneous rotation vector of Darboux trihedron as

$$w = -\frac{t}{T_g} - \frac{g}{R_n} + \frac{N}{R_g}\ . \tag{2.17}$$

The formulae (2.16) can be given by (2.17) as follows:

$$\frac{dt}{ds} = w \wedge t\ ,\ \frac{dg}{ds} = w \wedge g\ ,\ \frac{dN}{ds} = w \wedge N\ . \tag{2.18}$$

Theorem 2.1. If the radius of torsion of the space-like curve (c) drawn on time-like surface $y = y(u,v)$ is T and the hyperbolic angle between the time-like unit vectors n and g is θ, then we have

$$\frac{1}{T_g} = \frac{1}{T} + \frac{d\theta}{ds}\ . \tag{2.19}$$

Proof. Since $\langle n,g\rangle = -\cosh\theta$ we have

$$\langle\frac{dn}{ds}\ ,\ g\rangle + \langle n\ ,\ \frac{dg}{ds}\rangle = -\sinh\theta\ \frac{d\theta}{ds}\ .$$

By (2.3) and (2.16) we obtain

$$(\frac{1}{R}\ t + \frac{1}{T}\ b)\, g + n\ (\frac{1}{R_g}\ t + \frac{1}{T_g}\ N) = -\sinh\theta\ \frac{d\theta}{ds}$$

$$\frac{1}{R}\ \langle t,g\rangle + \frac{1}{T}\ \langle b,g\rangle + \frac{1}{R_g}\ \langle n,t\rangle + \frac{1}{T_g}\ \langle n,N\rangle = -\sinh\theta\ \frac{d\theta}{ds}\ .$$

$$\frac{1}{T}\sinh\theta - \frac{1}{T_g}\sinh\theta = -\sinh\theta\,\frac{d\theta}{ds}$$

$$\frac{1}{T_g} = \frac{1}{T} + \frac{d\theta}{ds}\ .$$

Theorem 2.2. Let the radius of curvature of the space-like curve (c) on the time-like surface y(u,v) is R and the hyperbolic angle between the normal N of the surface and the binormal b is θ. If the radii of normal and geodesic curvatures are R_n and R_g respectively, then we have

$$\frac{1}{R_n} = \frac{\sinh\theta}{R}\ ,$$

$$\frac{1}{R_g} = \frac{\cosh\theta}{R}\ . \tag{2.20}$$

Proof. By (2.5) and (2.18) we have

$$(f - w) \wedge t = 0.$$

If the values of vectors f and w are written, we find

$$\frac{n}{R} = -\frac{N}{R_n} + \frac{g}{R_g}\ .$$

If the both sides of this equation are scalarly multiplied with the vectors N and g and considered the equalities

$$\langle N,g\rangle = 0\ ,\ \langle n,g\rangle = -\cosh\theta \text{ and } \langle g,g\rangle = -1$$

then the proof is completed.

3. THE FUNDAMENTAL THEOREMS CONNECTED WITH THE GEOMETRY OF TIME-LIKE SURFACES

a) Fundamental relations

We know from previous section that

$$N \wedge t = -g\ ,\ N \wedge t_1 = -g_1\ ,\ N \wedge t_2 = g_2.$$

In this situation, three Darboux trihedron are obtained as follows:

$$[t,g,N]\ ,\ [t_1,g_1,N] \text{ and } [t_2,g_2,N].$$

The Darboux instantaneous rotation vectors corresponding to this trihedrons are given by

$$w = \frac{t}{T_g} + \frac{g}{R_n} - \frac{N}{R_g}$$

$$w_1 = \frac{t_1}{(T_g)_1} + \frac{g_1}{(R_n)_1} - \frac{N}{(R_g)_1} \ , \quad w_2 = -\frac{t_2}{(T_g)_2} - \frac{g_2}{(R_n)_2} + \frac{N}{(R_g)_2} \ , \tag{3.1}$$

respectively.

Here, let us denote arc lengths of the curves (c_1), (c_2) and (c) measured in the certain direction from P as s_1, s_2 and s, respectively. In this case, the following formulae are written:

$$t_1 = \frac{y_u}{\|y_u\|} = \frac{y_u}{\sqrt{E}} \ , \ t_2 = \frac{y_v}{\|y_v\|} = \frac{y_v}{\sqrt{G}} \ , \ t = y_u \frac{du}{ds} + y_v \frac{dv}{ds} \ . \tag{3.2}$$

If we write the two initial terms of (3.2) in the third term, then we obtain

$$t = y_u \frac{du}{ds} + y_v \frac{dv}{ds} = \sqrt{E}\ t_1 \frac{du}{ds} + \sqrt{G}\ t_2 \frac{dv}{ds} \ . \tag{3.3}$$

Let θ be the hyperbolic angle between t and t_1. If we take the inner poroduct of the equation (3.3) with t_1 and t_2, then the following formulae can be written:

$$\langle t,t_1\rangle = -\cosh\theta = -\sqrt{E}\frac{du}{ds} \ , \ \langle t,t_2\rangle = \sinh\theta = \sqrt{G}\frac{dv}{ds} \tag{3.4}$$

$$t = \cosh\theta\ t_1 + \sinh\theta\ t_2 \ . \tag{3.5}$$

For arcs ds, ds_1 and ds_2

$$ds^2 = Edu^2 - Gdv^2 \ ; \ F = 0$$

$$ds_1^2 = Edu^2 \ ; \ v = \text{const.}, \ dv = 0 \ ; \ ds_2^2 = Gdv^2 \ ; \ u = \text{const.}, \ du = 0 \ . \tag{3.6}$$

If the formulae (3.4) and (3.6) are compared, then we have

$$\cosh\theta = \frac{\sqrt{E}du}{ds} = \frac{ds_1}{ds} \ , \ \sinh\theta = \frac{\sqrt{G}dv}{ds} = \frac{ds_2}{ds} \ . \tag{3.7}$$

Furthermore, since $t \wedge N = g$ we can write

$$g = \cosh\theta\ g_1 - \sinh\theta\ g_2. \tag{3.8}$$

Instead of considering the time-like curve (c), if we take space-like curve (c) perpendicular to this curve then we can write

$$t = \sinh\theta\ t_1 + \cosh\theta\ t_2 \tag{3.9}$$

$$g = -\sinh\theta\ g_1 - \cosh\theta\ g_2. \tag{3.10}$$

The proof of fundamental theorem.

Let us choose the parameter curves as curvature lines. The tangent directions of (c_1) and (c_2) at the point P are given by

$$t_1 = \frac{y_u}{\sqrt{E}}\ ,\ t_2 = \frac{y_v}{\sqrt{G}}\ .$$

Furthermore, from the O we can write. Rodriques formulae

$$N_u = \frac{1}{R_1}\sqrt{E}\ t_1\ ,\ N_v = \frac{1}{R_2}\sqrt{G}\ t_2\ .$$

The direction of an arbitrary tangent is

$$t = y_u \frac{du}{ds} + y_v \frac{dv}{ds} = \sqrt{E}\ t_1 \frac{du}{ds} + \sqrt{G}\ t_2 \frac{dv}{ds}\ .$$

By (3.4) we obtain

$$\frac{dN}{ds} = \frac{\cosh\theta}{R_1} t_1 + \frac{\sinh\theta}{R_2} t_2\ . \tag{3.11}$$

On the other hand, if we consider the equalities (2.7) and (3.8) we can write

$$\frac{dN}{ds} = \left(\frac{\cosh\theta}{R_n} - \frac{\sinh\theta}{T_g}\right) t_1 + \left(\frac{\sinh\theta}{R_n} - \frac{\cosh\theta}{T_g}\right) t_2\ . \tag{3.12}$$

By (3.11) and (3.12) the proof is completed.

The fundamental formulae of the theory of time-like surfaces can be given as the consequences of the formula (1.1):

Theorem 3.1. If the normal curvature corresponding to perpendicular two directions taken on the time-like surface are $\frac{1}{(R_n)_1}$ and $\frac{1}{(R_n)_2}$, and also the geodesic torsions corresponding to this directions are $\frac{1}{(T_g)_1}$ and $\frac{1}{(T_g)_2}$ then the Gaussian curvature is given by

$$K = \frac{1}{R_1 R_2} = \frac{1}{(R_n)_1 (R_n)_2} + \frac{1}{(T_g)_1 (T_g)_2}\,. \tag{3.13}$$

Proof. Instead of considering the tangent direction t of the time-like curve (c) if we consider the tangent direction t_1 of the curve (c_1) we can write

$$\frac{\cosh\theta}{R_1} = \frac{\cosh\theta}{(R_n)_1} - \frac{\sinh\theta}{(T_g)_1}$$
$$\frac{\sinh\theta}{R_2} = \frac{\sinh\theta}{(R_n)_1} - \frac{\cosh\theta}{(T_g)_1}\,. \tag{3.14}$$

Similarly, if we consider the tangent direction t_2 of the curve (c_2) we obtain

$$\frac{\sinh\theta}{R_1} = \frac{\sinh\theta}{(R_n)_2} + \frac{\cosh\theta}{(T_g)_2}$$
$$\frac{\cosh\theta}{R_2} = \frac{\cosh\theta}{(R_n)_2} + \frac{\sinh\theta}{(T_g)_2}\,. \tag{3.15}$$

By the equalities (3.14) and (3.15) the value of K is found.

Theorem 3.2. Let us assume that the derivatives of unit vectors of the Darboux trihedrons [t,g,N], $[t_1,g_1,N]$ and $[t_2,g_2,N]$ with respect to parameter are exist and continuous. In this case, the following equalities are satisfied:

$$\frac{dt_1}{ds} = \begin{vmatrix} \cosh\theta & \sinh\theta \\ -\frac{1}{(T_g)_2} & \frac{1}{(R_n)_1} \end{vmatrix} N - \begin{vmatrix} \cosh\theta & \sinh\theta \\ \frac{1}{(R_g)_2} & \frac{1}{(R_g)_1} \end{vmatrix} t_2\,,$$
$$\frac{dt_2}{ds} = - \begin{vmatrix} \sinh\theta & \cosh\theta \\ \frac{1}{(T_g)_1} & \frac{1}{(R_n)_2} \end{vmatrix} N - \begin{vmatrix} \cosh\theta & \sinh\theta \\ \frac{1}{(R_g)_2} & \frac{1}{(R_g)_1} \end{vmatrix} t_1\,. \tag{3.16}$$

Proof. From the (3.5) and (3.8) we know that

$$t = \cosh\theta\ t_1 + \sinh\theta\ t_2,$$

$$g = -\sinh\theta\ g_1 - \cosh\theta\ g_2.$$

Furthermore we have $g_1 = -t_2$, $g_2 = t_1$. On the other hand, we can write

$$\begin{aligned} \frac{dt_1}{ds} &= \frac{\partial t_1}{\partial s_1}\cosh\theta + \frac{\partial g_2}{\partial s_2}\sinh\theta \\ \frac{dt_2}{ds} &= -\frac{\partial g_1}{\partial s_1}\cosh\theta + \frac{\partial t_2}{\partial s_2}\sinh\theta . \end{aligned} \tag{3.17}$$

If we consider the Darboux derivative formulae given for the curves (c_1) and (c_2), we obtain

$$\begin{aligned} \frac{\partial t_1}{\partial s_1} &= \frac{1}{(R_g)_1} g_1 + \frac{1}{(R_n)_1} N , \frac{\partial g_1}{\partial s_1} = \frac{1}{(R_g)_1} t_1 + \frac{1}{(T_g)_1} N , \\ \frac{\partial t_2}{\partial s_2} &= \frac{1}{(R_g)_2} g_2 - \frac{1}{(R_n)_2} N , \frac{\partial g_2}{\partial s_2} = \frac{1}{(R_g)_2} t_2 + \frac{1}{(T_g)_2} N . \end{aligned} \tag{3.18}$$

If this derivatives are replacing in (3.17) the proof is completed.

Corollary 3.3. If we consider the formulae (3.18) then the Gaussian curvature is given by

$$K = \frac{1}{R_1 R_2} = \langle \frac{\partial t_1}{\partial s_1} , \frac{\partial t_2}{\partial s_2} \rangle + \langle \frac{\partial t_1}{\partial s_2} , \frac{\partial t_2}{\partial s_1} \rangle . \tag{3.19}$$

Corollary 3.4. The relation

$$\langle t_2 , \frac{dt_1}{ds} \rangle = \langle -t_1 , \frac{dt_2}{ds} \rangle = \frac{\cosh\theta}{(R_g)_1} - \frac{\sinh\theta}{(R_g)_2} \tag{3.20}$$

is valid.

Theorem 3.5. The normal curvatures take the maximum or minumum value at principal directions.

Proof. By the formula (1.1) we can write

$$\left(\frac{1}{R_1} - \frac{1}{R_n}\right)\left(\frac{1}{R_2} - \frac{1}{R_n}\right) = \frac{1}{T_g^2} > 0 . \tag{3.21}$$

This shows that $\frac{1}{R_1} - \frac{1}{R_n}$ and $\frac{1}{R_2} - \frac{1}{R_n}$ have the same signature. Without loss of generality, let us assume $\frac{1}{R_2} < \frac{1}{R_1}$. In this case, we obtain that $\frac{1}{R_n} < \frac{1}{R_2}$ and $\frac{1}{R_n} > \frac{1}{R_1}$. This completes the proof.

Theorem 3.6. (Enneper Formula) There exists the relation

$$\frac{1}{T_g} = \frac{1}{T} = \sqrt{K} \tag{3.22}$$

between the torsion of asymptotic lines and the Gaussian curvature of the time-like surface, at the point P.

Proof. On the asymptotic lines, we know that $\frac{1}{R_n} = 0$. If we consider the equation (3.21) the proof is completed.

Theorem 3.7. (O. Bonnet Formula) For the geodesic torsion, we can write

$$\frac{1}{T_g} = \sinh\theta \cosh\theta \left(\frac{1}{R_1} - \frac{1}{R_2}\right). \tag{3.23}$$

Proof. In the formula (1.1), if we multiply the first equation with sinhθ and the second equation with coshθ, and derive from first the second, the formula (3.23) is found.

Theorem 3.8. (Euler Formula) The normal curvature at any direction is given by

$$\frac{1}{R_n} = \frac{\cosh^2\theta}{R_1} - \frac{\sinh^2\theta}{R_2}, \tag{3.24}$$

where $\frac{1}{R_1}$ and $\frac{1}{R_2}$ are principal curvatures.

Proof. By the formula (1.1) we have

$$\begin{aligned} \frac{\cosh^2\theta}{R_1} &= \frac{\cosh^2\theta}{R_n} - \frac{\sinh^2\theta \cosh^2\theta}{T_g}, \\ \frac{\sinh^2\theta}{R_2} &= \frac{\sinh^2\theta}{R_n} + \frac{\sinh\theta \cosh^2\theta}{T_g}. \end{aligned} \tag{3.25}$$

If we derive from first equation of (3.25) the formula (3.24) is obtained.

Theorem 3.9. (J. Liouville Formula) The geodesic curvature of an arbitrary time-like curve (c) at direction t is given by

$$\frac{1}{R_g} = \frac{\cosh\theta}{(R_g)_1} - \frac{\sinh\theta}{(R_g)_2} - \frac{d\theta}{ds}, \tag{3.26}$$

where $\frac{1}{(R_g)_1}$ and $\frac{1}{(R_g)_2}$ are geodesic curvatures at the directions t_1 and t_2, respectively.

Proof. Differentiating the equation (3.5) with respect to s, we obtain

$$\frac{dt}{ds} = \left(t_1 \sinh\theta + t_2 \cosh\theta\right)\frac{d\theta}{ds} + \cosh\theta \frac{dt_1}{ds} + \sinh\theta \frac{dt_2}{ds} .$$

By (2.5) and (2.18) we can write

$$\frac{1}{R_g} g - \frac{1}{R_n} N = -g \frac{d\theta}{ds} + \cosh\theta \frac{dt_1}{ds} + \sinh\theta \frac{dt_2}{ds} .$$

From this equality, we find that

$$\frac{1}{R_g} = -\frac{d\theta}{ds} + \langle g , \cosh\theta \frac{dt_1}{ds} + \sinh\theta \frac{dt_2}{ds}\rangle ,$$

$$\frac{1}{R_s} = -\frac{d\theta}{ds} + \langle -\sinh\theta\, t_1 - \cosh\theta\, t_2 , \cosh\theta \frac{dt_1}{ds} + \sinh\theta \frac{dt_2}{ds}\rangle ,$$

$$\frac{1}{R_s} = -\frac{d\theta}{ds} - \cosh^2\theta \langle\frac{dt_1}{ds} , t_2\rangle - \sinh^2\theta \langle\frac{dt_2}{ds} , t_1\rangle ,$$

$$\frac{1}{R_s} = -\frac{d\theta}{ds} - (\cosh^2\theta - \sinh^2\theta) \langle t_2 , \frac{dt_1}{ds}\rangle ,$$

$$\frac{1}{R_s} = -\frac{d\theta}{ds} - \langle t_2 , \frac{dt_1}{ds}\rangle .$$

If we consider the formula (3.19) the proof is completed.

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