QUASI - HADAMARD PRODUCT OF p - VALENT FUNCTIONS

M.K. AOUF., A. SHAMANDY and M.F. YASSEN

Department of Mathematics Faculty of Science University of Mansoura Mansoura, EGYPT.

ABSTRACT

The authors establish certain results concerning the quasi–Hadamard product of analytic and p–valent functions with negative coefficients analogous to the results due to Vinod Kumar.

1. INTRODUCTION

Let $S_p(\alpha, \beta, \lambda)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in N = \{1, 2, \ldots\}) \tag{1.1}$$

which are analytic and p–valent in the unit disc $U = \{z: |z| < 1\}$ and satisfy the condition

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - p}{\alpha \dfrac{zf'(z)}{f(z)} + p - \lambda(\alpha+1)} \right| < \beta \tag{1.2}$$

for some α $(0 \leq \alpha \leq 1)$, β $(0 < \beta \leq 1)$, λ $(0 \leq \lambda < p)$ and for all $z \in U$. The class $S_p(\alpha, \beta, \lambda)$ was studied by Owa and Aouf [4].

Throughout the paper, let the functions of the form

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_p > 0,\ a_{p+n} \geq 0,\ p \in N), \tag{1.3}$$

$$f_i(z) = a_{p,i} z^p - \sum_{n=1}^{\infty} a_{p+n,i} z^{p+n} \quad (a_{p,i} > 0,\ a_{p+n,i} \geq 0,\ p \in N), \tag{1.4}$$

$$g(z) = b_p z^p \quad - \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \ (b_p > 0, \ b_{p+n} \geq 0, \ p \in N), \qquad (1.5)$$

and

$$g_j(z) = b_{p,j} z^p \quad - \sum_{n=1}^{\infty} b_{p+n,j} z^{p+n} \ (b_{p,j} > 0, \ b_{p+n,j} \geq 0, \ p \in N), \qquad (1.6)$$

be analytic and p–valent in U.

Let $S^*_p (\alpha, \beta, \lambda)$ denote the class of functions $f(z)$ of the form (1.3) and satisfying (1.2) for some α, β, λ and for all $z \in U$. Also let $C^*_p (\alpha, \beta \ \lambda)$ denote the class of functions of the form (1.3) such that

$$\frac{zf'(z)}{p} \in S^*_p (\alpha, \beta, \lambda).$$

We note that when $a_p = \alpha = \beta = 1$, the classes $S^*_p (1, 1, \lambda) = T^* (p, \lambda)$ and $C^*_p (1, 1, \lambda) = C (p, \lambda)$ were studied by Owa [3].

Using similar arguments as given by Owa [3] we can easily prove the following analogous results for functions in the classes $S^*_p (\alpha, \beta, \lambda)$ and $C^*_p (\alpha, \beta, \lambda)$.

A function $f(z)$ defined by (1.3) belongs to the class $S^*_p (\alpha, \beta, \lambda)$ if and only if

$$\sum_{n=1}^{\infty} [\{n (1 + \alpha\beta) + \beta (1 + \alpha) (p-\lambda)\} a_{p+n}] \leq \beta (1 + \alpha) (p-\lambda) a_p \qquad (1.7)$$

and $f(z)$ defined by (1.3) belongs to the class $C^*_p (\alpha, \beta, \lambda)$ if and only if

$$\sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p} \right) \{n (1 + \alpha\beta) + \beta (1 + \alpha) (p-\lambda)\} a_{p+n} \right] \leq \beta (1 + \alpha) (p-\lambda) a_p.$$

$$(1.8)$$

We now introduce the following class of analytic and p–valent functions which plays an important role in the discussion that follows:

A function $f(z)$, defined by (1.3), belongs to the class $S^*_{p, k} (\alpha, \beta, \lambda)$ if and only if

$$\sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p}\right)^k \{n\,(1\,+\,\alpha\beta)\,+\,\beta\,(1\,+\,\alpha)\,(p-\lambda)\}\,a_{p+n} \right] \leq \beta\,(1+\,\alpha)\,(p-\lambda)\,a_p,$$

$$(1.9)$$

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $0 \leq \lambda < p$ and k is any fixed non-negative real number.

We note that, for every nonnegative real number k, the class $S^*_{p,\,k}\,(\alpha,\,\beta,\,\lambda)$ is nonempty as the functions of the form

$$f(z)\,=\,a_p z^p\,-\,\sum_{n=1}^{\infty}\,\frac{\beta\,(1\,+\,\alpha)\,(p-\lambda)\,a_p}{\left(\dfrac{p+n}{p}\right)^k \{n\,(1+\alpha\beta)+\beta(1+\alpha)(p-\lambda)\}}\,\lambda_{p+n} z^{p+n},\,(1.10)$$

where $a_p > 0$, $\lambda_{p+n} \geq 0$ and $\sum_{n=1}^{\infty} \lambda_{p+n} \leq 1$, satisfy the inequality (1.9).

It is evident that $S^*_{p,\,1}\,(\alpha,\,\beta,\,\lambda)\,=\,C_p^{*}\,(\alpha,\,\beta,\,\lambda)$ and, for $k=0$, $S^*_{p,\,k}\,(\alpha,\,\beta,\,\lambda)$ is identical to $S^*_p\,(\alpha,\,\beta,\,\lambda)$. Further, $S^*_{p,\,k}\,(\alpha,\,\beta,\,\lambda) \subset S^*_{p,\,k}\,(\alpha,\,\beta,\,\lambda)$ if $k > h \geq 0$, the containment being proper. Whence, for any positive integer k, we have the inclusion relation

$$S^*_{p,\,k}\,(\alpha,\,\beta,\,\lambda) \subset S^*_{p,\,k-1}\,(\alpha,\,\beta,\,\lambda)\ \subset\ \dots\ \subset\ S^*_{p,\,2}\,(\alpha,\,\beta,\,\lambda) \subset$$
$$C^*_p\,(\alpha,\,\beta,\,\lambda) \subset S^*_p\,(\alpha,\,\beta,\,\lambda).$$

Let us define the quasi–Hadamard product of the functions f(z) and g(z) by

$$f_*g\,(z)\,=\,a_p b_p z^p\,-\,\sum_{n=1}^{\infty}\,a_{p+n} b_{p+n} z^{p+n}.\qquad(1.11)$$

Similarly, we can define the quasi–Hadamard product of more then two functions.

In this paper we establish certain results concerning the quasi–Hadamard product of functions in the classes $S^*_{p,\,k}\,(\alpha,\,\beta,\,\lambda)$, $S^*_p\,(\alpha,\,\beta,\,\lambda)$ and $C^*_p\,(\alpha,\,\beta,\,\lambda)$ analogous to the results due to Vinod Kumar [1, 2].

2. THE MAIN THEOREMS

Theorem 1. Let functions $f_i(z)$ defined by (1.4) be in the class $C^*_p\,(\alpha,\,\beta,\,\lambda)$ for every $i = 1, 2,\dots, m$; and let the functions $g_j(z)$

defined by (1.6) be in the class $S^*_p(\alpha, \beta, \lambda)$ for every $j = 1, 2, \ldots, q$. Then, the quasi–Hadamard product $f_1 * f_2 * \ldots * f_m * g_1 * g_2 * \ldots * g_q(z)$ belongs to the class $S^*_{p,\, 2m+q-1}(\alpha, \beta, \lambda)$.

Proof: We denote the quasi–Hadamard product $f_1 * f_2 * \ldots * f_m * g_1 * g_2 \ldots * g_q(z)$ by the function $g(z)$, for the sake of convenience.

Clearly,

$$h(z) = \left\{ \prod_{i=1}^{m} a_{p,\, i} \prod_{j=1}^{q} b_{p,\, j} \right\} z^p - \sum_{n=1}^{\infty} \left\{ \prod_{i=1}^{m} a_{p+n,\, i} \prod_{j=1}^{q} b_{p+n,\, j} \right\} z^{p+n}. \quad (2.1)$$

To prove the theorem, we need to show that

$$\sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p} \right)^{2m+q-1} \left\{ n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda) \right\} \left\{ \prod_{i=1}^{m} a_{p+n,\, i} \prod_{j=1}^{q} b_{p+n, j} \right\} \right]$$

$$\leq \beta(1+\alpha)(p-\lambda) \left(\prod_{i=1}^{m} a_{p,\, i} \prod_{j=1}^{q} b_{p,\, j} \right). \quad (2.2)$$

Since $f_i(z) \in C^*_p(\alpha, \beta, \lambda)$, we have

$$\sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p} \right) \left\{ n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda) \right\} a_{p+n,\, i} \right] \leq \beta(1+\alpha)(p-\lambda) a_{p,\, i}, \quad (2.3)$$

for every $i = 1, 2, \ldots, m$. Therefore

$$\left(\frac{p+n}{p} \right) \left\{ n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda) \right\} a_{p+n,\, i} \leq \beta(1+\alpha)(p-\lambda) a_{p,\, i}$$

or

$$a_{p+n,\, i} \leq \left[\frac{\beta(1+\alpha)(p-\lambda)}{\left(\dfrac{p+n}{p} \right) \left\{ n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda) \right\}} \right] a_{p,\, i},$$

for every $i = 1, 2, \ldots, m$. The right–hand expression of this last inequality is not greater than $\left(\dfrac{p+n}{p} \right)^{-2} a_{p,\, i}$. Hence

$$a_{p+n, \, i} \leq \left(\frac{p+n}{p} \right)^{-2} a_{p, \, i}, \tag{2.4}$$

for every $i = 1, 2, \ldots, m$. Similarly, for $g_j(z) \in S^*_p (\alpha, \beta, \lambda)$, we have

$$\sum_{n=1}^{\infty} \, [\{ n (1+\alpha\beta) + \beta (1 + \alpha) (p-\lambda) \} \, b_{p+n, \, j}] \leq \beta (1+\alpha) (p-\lambda) \, b_{p, \, j} \tag{2.5}$$

for every $j = 1, 2, \ldots, q$. Whence we obtain

$$b_{p+n, \, j} \leq \left(\frac{p+n}{p} \right)^{-1} b_{p, \, j}, \tag{2.6}$$

for every $j = 1, 2, \ldots, q$.

Using (2.4) for $i = 1, 2, \ldots, m$, (2.6) for $j = 1, 2, \ldots, q-1$, and (2.5) for $j = q$, we get

$$\sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p} \right)^{2m+q-1} \{ n (1+\alpha\beta) + \beta (1+\alpha) (p-\lambda) \} \left\{ \prod_{i=1}^{m} a_{p+n, \, i} \prod_{j=1}^{q} b_{p+n, \, j} \right\} \right]$$

$$\leq \sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p} \right)^{2m+q-1} \{ n (1+\alpha\beta) + \beta (1+\alpha) (p-\lambda) \} \, b_{p+n, \, q} \right.$$

$$\left. \cdot \left\{ \left(\frac{p+n}{p} \right)^{-2m} \left(\frac{p+n}{p} \right)^{-(q-1)} \prod_{i=1}^{m} a_{p, \, i} \prod_{j=1}^{q-1} b_{p, \, j} \right) \right]$$

$$= \sum_{n=1}^{\infty} \, [\{ n (1+\alpha\beta) + \beta (1 + \alpha) (p-\lambda) \} \, b_{p+n, \, q}] \left(\prod_{i=1}^{m} a_{p, \, i} \prod_{j=1}^{q-1} b_{p, \, j} \right)$$

$$\leq \beta (1+\alpha) (p-\lambda) \left(\prod_{i=1}^{m} a_{p, \, i} \prod_{j=1}^{q} b_{p, \, j} \right).$$

Hence $h(z) \in S^*_{p, \, 2m+q-1} (\alpha, \beta, \lambda)$. This completes the proof.

We note that the required estimate can also be obtained by using (2.4) for $i = 1, 2, \ldots, m-1$, (2.6) for $j = 1, 2, \ldots, q$, and (2.3) for $i=m$.

Now we discuss the applications of Theorem 1. Taking into account the quasi-Hadamard product of the functions $f_1(z)$, $f_2(z)$, $\ldots$, $f_m(z)$ only, in the proof of Theorem 1, and using (2.4) for $i = 1, 2, \ldots$, $m-1$ and (2.3) for $i = m$, we are led to

Corollary 1. Let the fuctions $f_i(z)$ defined by (1.4) belong to the class $C^*_p(\alpha, \beta, \lambda)$ for every $i = 1, 2, \ldots, m$. Then the quasi–Hadamard product $f_1*f_2*\ldots*f_m(z)$ belongs to the class $S^*_{p, 2m-1}(\alpha, \beta, \lambda)$.

Next, taking into account the quasi–Hadamard product of the functions $g_1(z)$, $g_2(z), \ldots, g_q(z)$ only, in the proof of Theorem 1, and using (2.6) for $j = 1, 2, \ldots, q-1$ and (2.5) for $j = q$, we are led to.

Corollary 2. Let the functions $g_j(z)$ defined by (1.6) belong to the class $S^*_p(\alpha, \beta, \lambda)$ for every $j = 1, 2, \ldots, q$. Then, the quasi–Hadamard product $g_1*g_2*\ldots*g_q(z)$ belongs to the class $S^*_{p, q-1}(\alpha, \beta, \lambda)$.

REFERENCES

[1] VINOD, KUMAR., Hadamard product of certain starlike functions. II, J. Math. Anal. Appl. 113 (1986), 230–234.

[2] VINOD, KUMAR., Quasi–Hadamard product of certain univalent functions, J. Math. Anal. Appl. 126 (1987), 70–77.

[3] S. OWA., On certain class of p–valent functions with negative coefficients, Simon Stevin 59 (1985), no. 4, 385–402.

[4] S. OWA., and M.K. AOUF., On subclasses of p–valent functions starlike in the unit disc, J. Fac. Sci. Tech. Kinki Univ. 24 (1988), 23–27.