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ON THE SHEAF H_n OF HIGHER HOMOTOPY GROUPS AS AN ABELIAN COVERING SPACE

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ABSTRACT

Let X be a connected and locally path connected topological space. Constructing the sheaf H_n of higher homotopy groups on X, its some characterizations are examined. Also, it is shown that H_n is a regular covering space as a sheaf of abelian groups. Finally, it is given "General Lifting Theorem" for the sheaf H_n and constructing the Quotient Sheaf $Q_{H'_n}$ for any group subsheaf H'_n of the sheaf H_n, it is shown that $Q_{H'_n}$ is a covering space as a sheaf of abelian groups.

1. INTRODUCTION

Let X be a connected and locally path connected topological space. Then, X is a path connected. For an arbitrary fixed point $c \in X$, we will consider X as a pointed topological space (X, c) unless otherwise stated. Let x be any point of X and $\pi_n(X, x)$ be higher homotopy groups of X with respect to x and

$$H_n = \bigvee_{x \in X} \pi_n(X, x) .$$

Clearly, H_n is a set over X and the mapping $\Psi: H_n \to X$ defined by $\Psi(\sigma_x) = x$ for any $\sigma_x \in (H_n)_x \subset H_n$, is an onto projection.

We introduce on H_n a natural topology as follows: Let x_0 an arbitrary fixed point of X, $W = W(x_0)$ be a path connected open neighborhood of x_0 and $\sigma_{x_0} = [\alpha]_{x_0}$ be a homotopy class of $(H_n)_{x_0}$. Since X path connected, there exists a path γ with initial point x_0 and with terminal point x, for every $x \in W$. Therefore, the path γ determines an isomorphism $\gamma^* :(H_n)_{x_0} \to (H_n)_x$ defined by $\gamma^*([\alpha]_{x_0}) = [\beta]_x$ for any $[\alpha]_{x_0} \in (H_n)_{x_0} \subset H_n$. Let us now define a mapping $s : W \to H_n$ such that

$s(x) = \gamma^*([\alpha]_{x_0}) = [\beta]_x$ for every $x \in W$. If $c \in W$, then we define $s(c) = \gamma^*([\alpha]_c) = [\alpha]_c$, by taking $[\gamma] = [1] \in (H_n)_c$. It is seen that, the mapping s depends on both the homotopy classes $[\alpha]_{x_0}$ and $[\gamma]$. Suppose that the homotopy class $[\gamma]$ is choosen as arbitrary fixed, for each $x \in W$. So, the mapping s depends on only the homotopy class $[\alpha]_{x_0}$. s is well-defined and $\Psi o s = 1_W$. Let us denote the totality of the mapping s defined over W by $\Gamma(W, H_n)$.

Let B a basis of path conneted open neighborhoods for each $x \in X$. Then,

$$T_n = \{s(W) : W \in B,\ s \in \Gamma(W, H_n)\}$$

is a topology - base on H_n[4, 10]. In this topology the mapping Ψ and s are continuous. Moreover Ψ is a local topological mapping and the mapping s is a local invers of Ψ. Because;

1. Let $\sigma_{x_0} \in H_n$. Then $\Psi(\sigma_{x_0}) = x_0 \in X$. If $W = W(x_0)$ is an open set, then $W = \bigcup_{i \in I} W_i$, where each $W_i \in B$. So, for each W_i, there exists a mapping $s_i : W_i \to H_n$ such that $\Psi o s_i = 1_{W_i}$ and $s_i(W_i) \in T_n$.

Let us now define a mapping $s : W \to H_n$ such that $s|W_i = s_i$, for each W_i. Thus

$$s(W) = \bigcup_{i \in I} s_i(W_i)$$

is an open set in H_n and $\Psi o s = 1_W$. Write $s(W) = U$. Since $\Psi o s = 1_W$, $so\Psi = 1_U$, then $\Psi|U : U \to W$ is bijective and $(\Psi|U)^{-1} = s$.

2. The topologies on U and W are subspace topologies obtained from H_n and X, respectively. Let $W' \subset W$ be an open set. It can be written that

$$W' = \bigcup_{i \in I} W'_i$$

such that $W'_i = W'_i \cap W'$ for any $i \in I$. Now, if we define a mapping

$$s'_i : W'_i \to U$$

such that $s'_i = s_i|W'_i$, for each W'_i, then we can define a mapping

$$s' : W' \to U$$

such that $s'|W'_i = s'_i$. So,

$$s'(W') = \bigcup_{i \in I} s'_i(W'_i) \subset U$$

is an open set. Hence $\Psi|U$ is a continuous mapping. On the other hand, if $U' \subset U$ is an open set, then

$$U' = \bigcup_{i \in I} s'_i(W'_i)$$

Hence

$$\Psi(U') = \bigcup_{i \in I} W'_i \subset W$$

is an open set. Thus, the mapping $s : W \to U$ is continuous.

Therefore (H_n, Ψ) is a sheaf over X. It is called "the sheaf of higher homotopy groups" [6]. s is called a section over W and the set of totality of sections over W is $\Gamma(W, H_n)$. The $(H_n)_x = \pi_n(X, x)$ is called the stalk of the sheaf H_n for any $x \in X$. The group $(H_n)_x = \pi_n(X, x), n > 1$, is commutative for every $x \in X$. The set $\Gamma(W, H_n)$ is a group with pointwise multiplication operation. Thus, the operation.: $H_n \oplus H_n \to H_n$ is continuous for every stalk of H_n [1]. Hence, H_n is a sheaf of abelian groups.

2. CHARACTERISTIC FEATURES OF H_n [3]

* Every section over an open set W can be extended to a section over X. In other words, the sections over W are the restrictions of the sections over X, i.e., $\Gamma(W, H_n) = \Gamma(s|W, H_n)$, $s \in \Gamma(X, H_n)$. A section over X is called a global section.

* All of the stalks of the sheaf H_n over X are isomorphic.

* Let $W \subset X$ be an open set and s_1, s_2 be any two sections in $\Gamma(W, H_n)$. If $s_1(x_0) = s_2(x_0)$ for any $x_0 \in W$, then $s_1(x) = s_2(x)$ for each $x \in W$.

* Let $W_1, W_2 \subset X$ be any two open sets in X, $W_1 \cap W_2 \neq \emptyset$ and $s_1 \in \Gamma(W_1, H_n)$, $s_2 \in \Gamma(W_2, H_n)$. If $s_1(x_0) = s_2(x_0)$ for any $x_0 \in W_1 \cap W_2$, then $s_1(x) = s_2(x)$, for every $x \in W_1 \cap W_2$.

3. THE SHEAF H_n AS A COVERING SPACE

Now, we shall prove that, H_n is a regular covering space of X.

Theorem 3.1. Let H_n be the sheaf of abelian groups over (X, c) and W be an open set in X. Then

$$(H_n)_c \cong \Gamma(W, H_n).$$

Proof. Let $W \subset X$ be an open set and $s \in \Gamma(W, H_n)$. Then, there exists a unique element $\sigma_c = [\alpha]_c \subset (H_n)_c$ such that

$$s(x) = \gamma^*([\alpha]_c) = [\beta]_x$$

for every $x \in W$. That is, to each element of $(H_n)_c$, there correspondence only one element in $\Gamma(W, H_n)$. Let us denote this correspondence by $\Phi : (H_n)_c \to \Gamma(W, H_n)$ such that $\Phi(\sigma_c) = s$ for any $\sigma_c \in (H_n)_c$. Let $\sigma_c^1 = [\alpha_1]_c$, $\sigma_c^2 = [\alpha_2]_c \in (H_n)_c$ and σ_c^1, σ_c^2 determine the sections $s_1, s_2 \in \Gamma(W, H_n)$, respectively. Then

$$s_1(x) = \gamma^*([\alpha_1]_c) = [\beta_1]_x$$

and

$$s_2(x) = \gamma^*([\alpha_2]_c) = [\beta_2]_x$$

for every $x \in W$. Then $s_1(x) \neq s_2(x)$, if $\sigma_c^1 \neq \sigma_c^2$. So Φ is one to one. As a result of the definition of Φ, Φ is onto. Thus Φ is a bijection.

Φ is a homomorphism. Because, if $\sigma_c^1 = [\alpha_1]_c$, $\sigma_c^2 = [\alpha_2]_c \in (H_n)_c$, then $\sigma_c^1.\sigma_c^2 = [\alpha_1\alpha_2]_c$. So the element $\sigma_c^1.\sigma_c^2 \in (H_n)_c$ defines a section $s \in \Gamma(W, H_n)$ such that

$$s(x) = (s_1.s_2)(x) = \gamma^*([\alpha_1.\alpha_2]_c) = [\beta_1.\beta_2]_x$$

for every $x \in W$. On the other hand for every $x \in W$,

$$\begin{aligned} s_1(x).s_2(x) &= \gamma^*([\alpha_1]_c).\gamma^*([\alpha_2]_c) \\ &= \gamma^*([\alpha_1]_c\,[\alpha_2]_c) \\ &= \gamma^*([\alpha_1.\alpha_2]_c) \\ &= [\beta_1.\beta_2]_x\,. \end{aligned}$$

Thus

$$\Phi(\sigma_c^1.\sigma_c^2) = s_1 s_2 = \Phi(\sigma_c^1).\ \Phi(\sigma_c^2)$$

Therefore, Φ is an isomorphism.

We can state as a results of Theorem 3.1. that, the stalk $(H_n)_c$ completely determines the group of sections over W. In particular, if we take W = X, then the stalk $(H_n)_c$ completely determines the group of global sections over X.

Now we can state the following corollary [2].

Corollary. Let H_n be the sheaf of abelian groups over X. $(H_n)_x$ be the stalk over the point $x \in X$ and $W = W(x)$ be an open set. Then, $(H_n)_x \cong \Gamma(W, H_n)$. Particularly, $(H_n)_x \cong \Gamma(X, H_n)$.

According to this corollary, we can say that, if $\sigma_x \in (H_n)_x$ is any element and $W = W(x)$ is an open set in X, then there is a unique section $s \in \Gamma(W, H_n)$ such that $s(x) = \sigma_x$. Since

$$\Psi|s(W) : s(W) \to W$$

is a topological mapping and $s = (\Psi|s(W))^{-1}$,

$$\Psi^{-1}(W) = \bigvee_{i \in I} s_i(W),\ s_i \in \Gamma(W, H_n)$$

and

$$\Psi|s_i(W) : s_i(W) \to W$$

is a topological mapping. So, the open set $W = W(x)$ is evenly covered by Ψ. Thus Ψ is a covering projection and (H_n, Ψ) is a covering space of X [7,8,9]. Moreover, (H_n, Ψ) is an abelian covering space of X.

Now, let $x_0 \in X$ be any point and γ be an arc with initial point x_0, Then, the mapping

$$s o \gamma : I \to H_n$$

is a continuous mapping and $\Psi \ o \ (s o \gamma) = \gamma$. If we write $(s o \gamma)(x_0) = \rho_{x_0} \in (H_n)_{x_0}$, then $s o \gamma$ is a lifting of γ from the initial point ρ_{x_0} over x_0 in H_n.

Write $so\gamma = \gamma^*$, then γ^* is unique, because the mapping $\Psi|s(X) : s(X) \to X$ is a homeomorphism.

We can then state the following theorem.

Theorem 3.2. Let (H_n, Ψ) be the sheaf of abelian groups over X, $x_0 \in X$ be any point and γ be a path with initial point x_0 in X. Then, γ has a unique lifting γ^* with initial point ρ_{x_0} in H_n, for $\rho_{x_0} \in (H_n)_{x_0}$.

Now, we give the following theorem.

Theorem 3.3. (Monodromy). Let (H_n, Ψ) be the sheaf of abelian groups over X and suppose that γ_1^* and γ_2^* are paths with common initial point ρ_{x_0} and terminal point ρ_{x_1} in H_n. Then, γ_1^* and γ_2^* are homotopic path in H_n if and only if $\Psi o\gamma_1^*$ and $\Psi o\gamma_2^*$ are homotopic paths in X.

Proof. If γ_1^* is homotopic to γ_2^* by a homotopy G, then ΨoG is a homotopy between $\Psi o\gamma_1^*$ and $\Psi o\gamma_2^*$. For a proof of the other half of the theorem, let x_0 and x_1 denote the common initial point and common terminal point $\Psi o\gamma_1^*$ and $\Psi o\gamma_2^*$, respectively. Let $H : I \times J \to X$ be a homotopy between $\Psi o\gamma_1^*$ and $\Psi o\gamma_2^*$. On the other hand, if $\rho_{x_0} \in (H_n)$, then there is a unique section $s \in \Gamma(X, H_n)$ such that $s(x_0) = \rho_{x_0}$. So,

$$so(\Psi o\gamma_1^*) = \gamma_1^*$$

and

$$so(\Psi o\gamma_2^*) = \gamma_2^*$$

Furthermore, soH is a homotopy between γ_1^* and γ_2^*.

Theorem 3.4. Let (H_n, Ψ) be the sheaf of abelian groups over X, $x_0 \in X$ be an arbitrary fixed point and $\rho_{x_0} \in (H_n)_{x_0}$ be any point. Then the fundamental group of H_n with respect to ρ_{x_0} is isomorphic to $(H_n)_{x_0}$.

From theorems 3.3. and 3.4, (H_n, Ψ) is a regular covering space of X.

Now, we give "General Lifting Theorem" for the sheaf H_n.

Theorem 3.5. Let $X = (X, x_0)$, $Y = (Y, y_0)$ be two connected and locally path connected topological space (or two Riemann spaces), (H_n, Ψ)

be the sheaf of abelian groups over the pointed topological space (X, x_0), $\rho_{x_0} \in \Psi^{-1}(x_0)$ be any point. If

$$f : (Y, y_0) \to (X, x_0)$$

be any continuous mapping, then f can be lifted to a unique continuous

$$f^* : (Y, y_0) \to (H_n, \rho_{x_0})$$

such that $\Psi o f = f^*$.

Proof. Let $f : (Y, y_0) \to (X, x_0)$ be a continuous mapping. Then $f(y_0) = x_0$. If $\rho_{x_0} \in \Psi^{-1}(x_0)$ any point, then there exists a unique section $s \in \Gamma(X, H_n)$ such that $s(x_0) = \rho_{x_0}$. Thus

$$sof : (Y, y_0) \to (H_n, \rho_{x_0})$$

is a continuous mapping and

$$\Psi o(sof) = f$$

So, sof is a lifting of f to H_n. Let us denote sof by f^*. f^* is unique, because the section s is unique.

We can now state the following theorem.

Theorem 3.6. Let $X = (X, x_0)$, $Y = (Y, y_0)$ be two connected and locally path connected topological space (or two Riemann Surfaces), (H_n, Ψ) be the sheaf of abelian groups over the pointed topological space (X, x_0), $\rho_{x_0} \in \Psi^{-1}(x_0)$ be any point and

$$f^*, g^* : (Y, y_0) \to (H_n, \rho_{x_0})$$

be any two continuous mappings such that $\Psi o f^* = \Psi o g^*$, then

$$f^* = g^* .$$

Proof. This is a result of Theorem 3.5.

4. SUBSHEAVES AND QUOTIENT SHEAVES OF H_n.

In this section, Constructing the Quotient sheaf $Q_{H'_n}$, for any subsheaf of the groups H'_n of the sheaf H_n, it is shown that $Q_{H'_n}$ is covering space as a sheaf of abelian groups.

We begin by giving the following definition [5].

Definition 4.1. Let H_n be the sheaf of abelian groups over X and $H'_n \subset H_n$ be an open set. Then H'_n is called a subsheaf of the sheaf H_n of abelian groups, if

i) $\Psi(H'_n) = X$

ii) For each point $x \in X$, the stalk $(H'_n)_x$ is a subgroup of $(H_n)_x$.

We now give the following theorem.

Theorem 4.1. (Existence Theorem). Let $X = (X, c)$ be a connected, locally path connected topological space and $(H_n)_c$ be higher homotopy group with respect to $c \in X$. Then each subset $(H'_n)_c$ of $(H_n)_c$ determines a sheaf over X.

As a result of Theorem 4.1.,

1. If $(H'_n)_c = (H_n)_c$, it is obtained that $H'_n = H_n$. So, the sheaves H'_n are subsheaves of H_n. Also, $\Psi' = \Psi|H'_n$.

2. If $(H'_{n_1})_c$, $(H_{n_2})_c$ are any two subset of $(H'_n)_c$ and $(H'_{n_1})_c \subset (H'_{n_2})_c$ then

$H'_{n_1} \subset H'_{n_2}$.

Furthermore, if $W \subset X$ is an open set, then

$\Gamma(W, H'_{n_1}) \subset \Gamma(W, H'_{n_2}) \subset \Gamma(W, H_n)$.

3. Let H'_n be a subsheaf of the sheaf H_n of abelian groups and $W \subset X$ be an open set. Then $\Gamma(W, H'_n)$ is a subgroup of $\Gamma(W, H_n)$. If we take $W = X$, then $\Gamma(X, H'_n)$ is a subgroup of $\Gamma(X, H_n)$.

Now, we give the following definition.

Definition 4.2. Let H_n be the sheaf of abelian groups over X and $H'_n \subset H_n$ be a subsheaf of abelian groups. Let us associate the set

$M_W = \Gamma(W, H_n)/\Gamma(W, H'_n)$

with the open set W, for each $W \subset X$ open. Then, the system $\{X, M_W, \gamma_{W,V}\}$ is a pre-sheaf [4]. The sheaf defined by the pre-sheaf $\{X, M_W, \gamma_{W,V}\}$ is called Quotient sheaf and it is denoted by $Q_{H'_n}$.

Theorem 4.2. Let H_n be the sheaf of abelian groups over X and $H'_n \subset H_n$ be a subsheaf of abelian groups. Then, the Quotient sheaf $Q_{H'_n}$ is a sheaf of abelian groups over X.

Proof. Let H_n be the sheaf of abelian groups over X and $H'_n \subset H_n$ be a subsheaf of abelian groups. Also, H'_n is a normal subsheaf of the sheaf H_n. So, $\Gamma(X, H'_n) \subset \Gamma(X, H_n)$ is a normal subgroup and $\Gamma(X, H_n)/(X, H'_n)$ is a group. Let

$$Q_{H'_n} = \bigvee_{x \in X} (Q_{H'_n})_x$$

and

$$(Q_{H'_n})_x = \{(W, [s])_x : W \subset X \text{ is an open set, } [s] \in \Gamma(X, H_n)/\Gamma(X, H'_n)\}.$$

So, the operation defined in each stalk $(Q_{H'_n})_x$ in the form of $(W, [s_1])_x \cdot (W,[s_2])_x = (W, [s_1 s_2])_x$ is well defined. It is easily seen that each stalk $(Q_{H'_n})_x$ is an abelian group with this operation for every $x \in X$. Since $(Q_{H'_n})_x \cong \Gamma(X, Q_{H'_n})$, $\Gamma(X, Q_{H'_n})$ is an abelian group. Thus, $Q_{H'_n}$ is a sheaf of abelian groups.

Moreover, $Q_{H'_n}$ is a covering space as a sheaf of abelian groups. Also, it is a regular covering space.

Theorem 4.3. Let H_n be the sheaf of abelian groups over X, $H'_n \subset H_n$ be a subsheaf of abelian groups and $Q_{H'_n}$ be quotient sheaf. Then the group $\Gamma(X, Q_{H'_n})$ is isomorphic to the quotient group $\Gamma(X, H_n)/\Gamma(X, H'_n)$.

Proof. To prove this theorem let us define the mapping

$$\gamma : \Gamma(X, H_n)/\Gamma(X, H'_n) \to \Gamma(X, Q_{H'_n})$$

in the form of $\gamma([s]) = \gamma[s]$, where γ representes inductive limit [4]. If $\gamma([s]) = 1$, then $\gamma[s] = 1$ and so, $\gamma[s](x) = (X, [e])_x$, for any $x \in X$. That is

$$(W, [s])_x = (W, [e])_x .$$

Thus,

$$[s] = [e] .$$

Hence, γ is one to one. Clearly γ is onto. Now, if $[s_1], [s_2] \in \Gamma(X, H_n)/\Gamma(X, H'_n)$ are any two elements, then

$$\begin{aligned}\gamma([s_1][s_2]) &= \gamma([s_1.s_2]) \\ &= \gamma[s_1.s_2] \\ &= \gamma[s_1].\gamma[s_2]\end{aligned}$$

Thus, γ is a homomorphism.

Therefore, $\gamma : \Gamma(X, H_n)/\Gamma(X, H'_n) \rightarrow \Gamma(X, Q_{H'_n})$ is an isomorphism.

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