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ON JACOBI'S THEOREMS

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ABSTRACT

In this paper, J. Jacobi's Theorems [9] have been considered for the spherical curves drawn on the unit dual sphere during the closed space motions. The integral invariants of the ruled surface corresponding, in the line space, to the spherical curve drawn by a fixed point on the moving unit dual sphere during the one-parameter closed motion were calculated with a different approach from the area vector used by H. R. Müller [11]. In addition, the ruled surfaces corresponding to the curves drawn by the unit tangent vector, principal normal vector or a unit vector on the osculating plane of the mentioned curve, were seen to be cones with this approach.

1. INTRODUCTION

J. Jacobi [9] showed that the indicator of tangents of any real closed spherical curve divides the surface area of a unit sphere into two equal parts. In the same paper, he also showed that the indicator of the principal normals of any closed curve also divides the surface area of the unit sphere into two equal parts. Then, W. Fenchel [2] and V.G. Avaqumovic [1], using Jacobi Theorems, showed that "A necessary and sufficient condition for a closed spherical indicator of the principal normals of another spatial curve is that the spherical curve divides the surface of the sphere into two equal parts" and "The spherical indicator of principal normals of a closed spherical curve divides the surface area of the unit sphere into equal parts". Also, Z. Yapar [16] showed that the spherical indicator of each unit vector lying in the osculating plane of a closed spherical curve which is fixed to the curve divides the surface area of the unit sphere into two equal parts. On the other hand, integral invariants of the ruled surfaces have been studied by many mathematicians. These, by name, are; H.H. Hacısalihoğlu [5], J. Hoschek [7], H.R. Müller [12], O. Gürsoy [4].

The angle of pitch and length of pitch, which are the real integral invariants of a closed ruled surface, are very important to study the geometry of lines from the perspectives of instantaneous space kinematics and mechanisms. For example, some theorems in the plane kinematics can be generalized to the ruled surfaces in the $\mathbb{R}^3$ and thus some relations between integral invariants of these ruled surfaces are given, in [4,5,6,7,8].

In this paper, the integral invariants of the ruled surface are studied with a different method considering the Jacobi Theorems with the help of area vector. Our aim is to present both a new way to study the geometry of the lines and bring new geometric comments. We hope that the presented results would bring new perspectives to the spatial kinematics.

2. BASIC CONCEPTS

A dual number has the form $a + \varepsilon a^*$, where a and a^* are real numbers and ε is the dual unit with the property $\varepsilon^2 = 0$. The set of all dual numbers is a commutative ring over the real numbers field and denoted by $\mathbb{D}$, [15]. The set $\mathbb{D}^3 = \{A = (A_1, A_2, A_3\}: A_i \in \mathbb{D}, 1 \le i \le 3\}$ is a module over the ring $\mathbb{D}$ which is called $\mathbb{D}$-module or dual space. We call the elements of $\mathbb{D}^3$ as dual vectors. A dual vector $\vec{A}$ may be written as $\vec{A} = \vec{a} + \varepsilon \vec{a}^*$, where $\vec{a}$ and $\vec{a}^*$ are real vectors in $\mathbb{R}^3$. The inner product of two dual vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\langle \vec{A}, \vec{B} \rangle = \langle \vec{a}, \vec{b} \rangle + \vec{\varepsilon} \left(\langle \vec{a}, \vec{b}^* \rangle + \langle \vec{a}^*, \vec{b} \rangle \right)$$

where $\langle \vec{a}, \vec{b} \rangle = \cos \varphi$ and $\langle \vec{a}, \vec{b}^* \rangle + \langle \vec{a}^*, \vec{b} \rangle = -\varphi^* \sin \varphi$, $0 \le \varphi \le \pi$. If $\vec{a} \ne \vec{0}$ the norm of $\vec{A}$ is defined by

$$\|\vec{A}\| = \left(\langle \vec{A}, \vec{A} \rangle \right)^{\frac{1}{2}} = \|\vec{a}\| + \varepsilon \frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|}$$

A dual vector $\vec{A}$ with norm $(1,0)$ is called a unit dual vector. The cross-product of dual vectors $\vec{A}$ and $\vec{B}$ is given by

$$\vec{A} \wedge \vec{B} = \vec{a} \wedge \vec{b} + \varepsilon \left(\vec{a} \wedge \vec{b}^* + \vec{a}^* \wedge \vec{b} \right).$$

The dual angle between the unit dual vectors $\vec{A}$ and $\vec{B}$ is given.

$$\langle \vec{A}, \vec{B} \rangle = \cos \Phi = \cos \varphi - \varepsilon\varphi^* \sin \varphi$$

where $\Phi = \varphi + \varepsilon\varphi^*$, $0 \le \varphi \le \pi$, $\varphi^* \in \mathbb{R}$, is a dual number. The real numbers φ and φ^* are the angle and the minimal distance between the two lines $\vec{A}$ and $\vec{B}$, respectively [3]. The geometric place of the points satisfying the equality $\|\vec{A}\| = (1,0)$ to be $A \neq (0,0^*)$ is called a unit dual sphere in ID-module [13]. E. Study gave the following theorem.

Theorem 2.1. (E. Study). There is a one to one mapping between the dual points of a unit dual sphere and the oriented lines in $\mathbb{R}^3$ [13].

According to the Theorem 2.1., the unit dual vector $\vec{A} = \vec{a} + \varepsilon\vec{a}^*$ corresponds to only oriented line; where the real vector $\vec{a}$ shows the direction of this line and the real vector $\vec{a}^*$ shows the vectorial moment of the unit vector $\vec{a}$ with respect to the origin point O.

Let us have a closed spherical dual curve (γ) of class C^2 on a unit dual sphere K' in $\mathbb{D}^3$. At the initial time, assume that the unit dual sphere K corresponding with K' to be K = K', where K' is a fixed sphere and K is a moving sphere with respect to K'. The curve (γ) decribes a closed dual spherical motion. Let us fixed the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ to any points T of the curve (γ). Here, $\vec{A}$, $\vec{B}$ and $\vec{C}$ are tangent, principal normal and binormal unit dual vectors, respectively. While drawing the closed dual spherical curve (γ) during the motion B = K/K', the end points of vectors on the unit dual sphere K' also draw closed spherical curves c(A), c(B) and c(C) respectively (Figure 1).

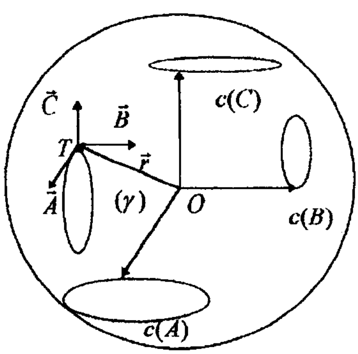

Fig. 1.

The differential of the frame $\left\{\vec{A},\vec{B},\vec{C}\right\}$ representing the moving sphere K are

$$\dot{\vec{A}} = \kappa\vec{B}$$

$$\dot{\vec{B}} = -\kappa\vec{A} + \tau\vec{C}$$

$$\dot{\vec{C}} = -\tau\vec{B}$$

The Darboux rotation vector of the motion is

$$\vec{Q} = \tau\vec{A} + \kappa\vec{C}\ ,\ Q = (\vec{q},\vec{q}^*)$$

Also the derivative vector $\dot{\vec{A}}$ in terms of Darboux rotation vector can be written as follows:

$$\dot{\vec{A}} = \vec{Q} \wedge \vec{A}$$

The curves and the surfaces correspond to the points and the curves on the unit dual sphere, respectively. Let A be a initial ruled surface corresponding to a constant line a = $(\vec{a},\vec{a}^*)$ of K during the motion B = K/K' in the lines space. For the initial ruled surface A an accompanying orthonormal trihedron [12] is

$$\vec{A} = (\vec{a},\vec{a}^*)$$

$$\vec{B} = \frac{\dot{\vec{A}}}{\|\dot{\vec{A}}\|} = \frac{\vec{Q} \wedge \vec{A}}{\sqrt{\langle\vec{Q},\vec{Q}\rangle - \left(\langle\vec{A},\vec{Q}\rangle\right)^2}}$$

$$\vec{C} = \vec{A} \wedge \vec{B} = \frac{\vec{Q} - \vec{A}\langle\vec{A},\vec{Q}\rangle}{\sqrt{\langle\vec{Q},\vec{Q}\rangle - \left(\langle\vec{A},\vec{Q}\rangle\right)^2}}$$

3. AREA VECTOR AND PROJECTION AREA

The dual angle of pitch of the closed ruled surface generated by a constant point X of K is

$$\Lambda_X = -\langle\vec{X},\vec{S}\rangle = \lambda_x - \varepsilon\ell_x \tag{3.1}$$

where λ_x and ℓ_x are the real integral invariants of this ruled surface and also $\vec{S}$ is the dual Steiner vector of the motion [4].

Let c(X) be the dual orbit, on K', of an arbitrary fixed dual point X on K. The dual spherical area F_X surrounded by the dual closed c(X) can be calculated as

$$F_X = 2\pi(1 - \nu) - \langle \vec{X}, \vec{S} \rangle \tag{3.2}$$

Here, ν is the rotation number of rotation of the centrode c(P) at the point X, $\vec{X}$ denotes the dual position vector of an arbitrary point of the dual closed curve c(X) on K' [5]

The area vector of the closed curve c(X) is given by

$$\vec{V}_X = \oint \vec{x}(t) \wedge \dot{\vec{x}}(t)\, dt \tag{3.3}$$

in [8]. On the other hand, the projection area of a planar region occurred by taking orthogonal projection onto a plane in the direction of a constant unit vector $\vec{n}$ of the curve c(X) is given by

$$2F_X^n = \langle \vec{n}, \vec{V}_X \rangle \tag{3.4}$$

in [8]. After these preparations we can give the following theorem:

Theorem 3.1. Let c(A), c(B) and c(C) be the spherical indicators of the unit dual vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, during the closed dual spherical motion B = K/K', respectively. The area vectors of these closed spherical indicators are

$$\begin{aligned} \vec{V}_A &= \vec{S} - \vec{A}\langle \vec{A}, \vec{S} \rangle \\ \vec{V}_B &= \vec{S} \\ \vec{V}_C &= \vec{A}\langle \vec{A}, \vec{S} \rangle \end{aligned} \tag{3.5}$$

where $\vec{S} = \oint \vec{Q}(t)\, dt$ is the dual Steiner rotation vector of the motion [10].

Corollary 3.2. The dual area vector $\vec{V}_B$ is equal to the sum of dual area vectors $\vec{V}_A$ and $\vec{V}_C$.

Corollary 3.3. The unit dual vector $\vec{B}$ is perpendicular to the area vectors $\vec{V}_A$ and $\vec{V}_C$.

If the expression (3.5) is separated into its real and dual parts, we have the following equalities:

$$\begin{aligned} \vec{V}_a &= \vec{s} + \vec{a}\lambda_a \quad , \quad & \vec{V}_a^* &= \vec{s}^* + \vec{a}^*\lambda_a - \vec{a}\ell_a \\ \vec{V}_b &= \vec{s} \quad , \quad & \vec{V}_b^* &= \vec{s}^* \\ \vec{V}_c &= -\vec{a}\lambda_a \quad , \quad & \vec{V}_c^* &= -\vec{a}^*\lambda_a + -\vec{a}\ell_a \end{aligned} \tag{3.6}$$

where $\vec{V}_a, \vec{V}_b, \vec{V}_c$ and $\vec{V}_a^*, \vec{V}_b^*, \vec{V}_c^*$ are real and dual area vectors, respectively. The dual angle of pitch of a ruled surface **A** corresponding to the closed dual spherical curve c(A) in the line space is

$$\Lambda_A = -\langle \vec{A}, \vec{V}_B \rangle = -\langle \vec{a}, \vec{V}_b \rangle - \varepsilon \left(\langle \vec{a}, \vec{V}_b^* \rangle + \langle \vec{a}^*, \vec{V}_b \rangle \right) \tag{3.7}$$

where, $\lambda_a = -\langle \vec{a}, \vec{V}_b \rangle = -\langle \vec{a}, \vec{V}_c \rangle$ and $\ell_a = \langle \vec{a}, \vec{V}_b^* \rangle + \langle \vec{a}^*, \vec{V}_b \rangle = \langle \vec{a}, \vec{V}_c^* \rangle + \langle \vec{a}^*, \vec{V}_c \rangle$ are the real angle of pitch and the real length of pitch of the ruled surface **A**.

4. ON JACOBI'S THEOREMS

A closed dual spherical curve (γ) can be written by means of the arc length-parameter [14]. When closed spherical curve (γ) is drawn by the position vector $\vec{r}$, its unit dual tangent vector $\vec{A}$, normal vector $\vec{B}$ and binormal vector $\vec{C}$ draw closed dual spherical curves.

Let c(A) be the spherical indicator of the unit dual vector $\vec{A}$ under the motion $B = K/K'$. If the area of the region surrounded by the curve c(A) denoted by F_A, then from eq. (3.2) and eq. (3.6), is

$$F_A = 2\pi(1 - \nu) + \lambda_a - \varepsilon\ell_a \tag{4.1}$$

Since the above area should be $F_A = 2\pi$ according to the Jacobi Theorem [1], we obtain

$$-2\nu\pi + \lambda_a - \varepsilon\ell_a = 0 \tag{4.2}$$

From (4.2), according to the equality of two dual numbers we have

$$\lambda_a = 2\nu\pi$$
$$\ell_a = 0 \tag{4.3}$$

From (3.1) we obtain the integral invariants of the ruled surface **B** corresponding to the spherical indicator c(B) of the unit dual vector $\vec{B}$, in the line space, as the following

$$\lambda_b = 0$$
$$\ell_b = 0 \tag{4.4}$$

If the area of the region surrounded by the curve c(B) denoted by F_B, then from eq. (3.2) and eq. (3.6), is

$$F_B = 2\pi(1 - \nu)$$

Since the above are should be $F_B = 2\pi$ according to the Jacobi Theorem [9], we obtain $\nu = 0$. Thus, we can give the following theorems:

Theorem 4.1. Let (γ) be a closed dual spherical curve on the unit dual sphere. Let **A** be the ruled surface corresponding to the spherical indicator of the tangent vector $\vec{A}$ of the dual curve (γ). The real angle of pitch and length of pitch, to be λ_a and ℓ_a respectively, we obtain

$$\lambda_a = -\langle\vec{a},\vec{V}_b\rangle = 2\nu\pi \ , \ \ell_a = \langle\vec{a},\vec{V}_b{}^*\rangle + \langle\vec{a}^*,\vec{V}_b\rangle = 0.$$

Theorem 4.2. Let (γ) be a closed dual spherical curve on the unit dual sphere. Let **B** be the ruled surface corresponding to the spherical indicator of the principal vector $\vec{B}$ of the dual curve (γ). The real angle of pitch and length of pitch of the ruled surface **B**, to be λ_b and ℓ_b respectively, $\lambda_b = 0$, $\ell_b = 0$. So it is a cone.

Further, by Study mapping, we have the following theorems:

Theorem 4.3. In the line space, the image of the closed spherical indicator of the tangent of a closed curve on the unit dual sphere under Study mapping is a cone which has an angle of pitch as

$$\lambda_a = -\langle\vec{a},\vec{V}_b\rangle = 2\nu\pi \ .$$

Theorem 4.4. In the line space, the image of the closed spherical indicator of principal normal of any closed curve on the unit dual sphere under Study mapping is a cone which has the angle of pitch $\lambda_b = 0$.

The number of rotation of the center P, which corresponds to drawing of the cone, does not depend on the curve (γ), and $\nu = 0$.

Let $\vec{C}$ be the binormal vector of the closed ruled surface (γ) and c(C) be the spherical indicator of the binormal vector $\vec{C}$. The area of the region surrounded by c(C) is

$$F_C = 2\pi(1 - \nu) + (\lambda_c - \varepsilon\ell_c)$$

In addition, the length of pitch is $\ell_c = \langle\vec{c},\vec{V}_b{}^*\rangle + \langle\vec{c}^*,\vec{V}_b\rangle$

Thus we can give the following theorem:

Theorem 4.5. In the line space, the spherical indicator of binormal of any closed dual spherical curve (γ), on the unit dual sphere, corresponds to a ruled surface. The length of pitch of this ruled surface only depends on the curve (γ), and $\ell_c = \langle\vec{c},\vec{V}_b{}^*\rangle + \langle\vec{c}^*,\vec{V}_b\rangle$

Now, let us consider all the unit dual vectors firmly attached to the curve which lies in the osculating plane of the closed curve (γ). Let $\vec{M}$ be one of these vectors. Let $\Theta = \theta + \varepsilon\theta^*$ be the angle between the unit dual vector $\vec{M}$ and the unit dual tangent vector $\vec{A}$. The vector $\vec{M}$ is written as follows:

$$\vec{M} = \cos\Theta\,\vec{A} + \sin\Theta\,\vec{B} \tag{4.5}$$

If the unit dual vector $\vec{M}$ is separated into its real or dual parts we obtain

$$\begin{aligned}\vec{m} &= \cos\theta\,\vec{a} + \sin\theta\,\vec{b}\\ \vec{m}^* &= \cos\theta\,\vec{a}^* + \sin\theta\,\vec{b}^* - \theta^*\sin\theta\,\vec{a} + \theta^*\cos\theta\,\vec{b}\end{aligned} \tag{4.6}$$

In the line space, let **M** be the ruled surface corresponding to the unit dual spherical indicator of the unit dual vector $\vec{M}$. The dual angle of pitch of this ruled surface, from eq. (4.3), eq. (4.5) and eq. (4.6), is obtained as follows;

$$\Lambda_M = -\langle \vec{M}, \vec{S} \rangle = \lambda_a \cos\theta - \varepsilon\, \lambda_a\, \theta^* \sin\theta = \lambda_a \cos\Phi$$

The real angle of pitch and the length of pitch of this ruled surface are, respectively,

$$\begin{aligned} \lambda_m &= \lambda_a \cos\theta \\ \ell_m &= \lambda_a\, \theta^* \sin\theta \end{aligned} \tag{4.7}$$

On the other hand, let c(M) be the closed spherical curve drawn by the unit dual vector $\vec{M}$ during the motion B = K/K'. The area of the region surrounded by the closed spherical curve c(M) is obtained as

$$F_M = 2\pi(1 - \nu) + \lambda_a \cos\Theta$$

Since this are should be 2π [16], we have

$$\lambda_a \cos\theta = 2\nu\pi\ ,\ \lambda_a\, \theta^* \sin\theta = 0.$$

By taking $0 < \theta < \frac{\pi}{2}$ and $\theta^* \neq 0$ we get

$$\lambda_a = 0 \tag{4.8}$$

So, we can give the following theorems:

Theorem 4.6. Let (γ) be a closed curve on the unit dual sphere. The real angle of pitch of the ruled surface corresponding to the dual tangent indicator of the closed curve (γ) is $\lambda_a = 0$. So it is a cone.

Theorem 4.7. Let $\vec{A}$ and $\vec{B}$ be the tangent and the principal normal vector of the closed curve (γ), respectively. The unit dual vector $\vec{A}$ is perpendicular to the area vector $\vec{V}_B$.

Substituting eq. (4.8) into eq. (4.7), we can give the following theorem:

Theorem 4.8. Let (γ) be a closed curve on the unit dual sphere. Let $\vec{M}$ be the unit dual vector firmly fixed to the curve which lies in the osculating plane of the closed curve (γ). The real angle of pitch and the length of pitch of the ruled surface corresponding to the spherical indicator of $\vec{M}$, in the line space, are $\lambda_m = 0$, $\ell_m = 0$ respectively.

Theorem 4.9. In the line space, the image of the spherical indicator of any unit dual vector $\vec{M}$ lying in the osculating plane of any closed curve (γ) drawn on unit dual sphere is a cone under Study mapping. The number of rotation of the center P, which corresponds to the drawing of the cone does not depend on the closed curve (γ).

Theorem 4.10. In the line space, the spherical indicator of tangents, principal normals and binormals of any closed dual spherical curve correspond to the ruled surface. The angle of pitch and the length of pitch these ruled surfaces does not depend on these indicators and the given closed spherical curve (γ).

Theorem 4.11. In the line space, closed curves, each of which divides the surface of the unit dual sphere into two equal are aparts, correspond to a cone.

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