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ON THE DUALITY OF GENERALISED EULER FORMULA FOR EUCLIDEAN HYPERSURFACES

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ABSTRACT

In order to define the generalised Euler formula in a dual manner, we studied the angles between two hyperplanes in R^{n+1} and we obtained that the Gauss curvature can be expressed by the normal curvature and its dual form.

I. INTRODUCTION

In Euclidean space R^{n+1} of dimension $n + 1$ we consider an n–dimensional hypersurface M given by a local coordinate system $\{u^1, u^2, \ldots, u^n\}$. Let $\{x_1, x_2, \ldots, x_{n+1}\}$ be an orthogonal coordinate system of R^{n+1}. We assume that the x_i's are C^∞– functions of u^α's and that $1 \leq i \leq n + 1$, $1 \leq \alpha \leq n$. Let X be a vector whose orthogonal components are $(x_1, \ldots, x_{n+1})$, then the hypersurface M can be characterized by a vector function

$$X = X(u^\alpha),\ \alpha = 1, \ldots, n. \tag{I.1}$$

Let us denote by N the unit normal vector field of the hypersurface M, then it satisfies the conditions $\langle N,N\rangle = 1$ and $\langle N, \frac{\partial X}{\partial u^\alpha}\rangle = 0$. Now let us introduce an orthonormal frame in R^{n+1} by e_i, and using this frame we can write that

$$N = \sum_{i=1}^{n+1} N_i\, e_i \tag{I.2}$$

and that

$$\frac{\partial X}{\partial u^k} = \sum_{i=1}^{n+1} (x_k)_i\, e_i,\ k = 1, \ldots, n, \tag{I.3}$$

where $N = N_i(u^\alpha)$, $\alpha = 1, \ldots, n$, $1 \leq i \leq n + 1$.

II. PRELIMINARIES

Let v denote a tangent vector of the tangent space $T_M(m)$ at the point m of hypersurface M. In this direction the curvature $\frac{1}{R}$ of the hypersurface M is defined by

$$\frac{1}{R} = - < v, \frac{\partial N}{\partial u^k} > = h_{\alpha\beta}\, u^\alpha\, u^\beta \tag{II.1}$$

where $h_{\alpha\beta}$ is the second fundamental tensor of M and defined as

$$h_{\alpha\beta} = < N, \frac{\partial^2 X}{\partial u^\alpha \partial u^\beta} > = - < \frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^\beta} > .$$

The principal curvatures at a point of M are the eigen values of the second fundamental tensor evaluated at this point. Hence they are the roots of the characteristic equation as follows

$$\det\left[h_{\alpha\beta} - \frac{1}{R} g_{\alpha\beta}\right] = (-1)^n \det(g_{\alpha\beta}) \left(\frac{1}{R} - \frac{1}{R_1}\right) \cdots \left(\frac{1}{R} - \frac{1}{R_n}\right)$$

$$= (-1)_n \det(g_{\alpha\beta}) \Big\{ \frac{1}{R^n} - \frac{1}{R^{n-1}}\left((\sum_{i_1=1}^{n} \frac{1}{R_{i_1}}\right) + \frac{1}{R^{n-2}}\left(\sum_{i_1 < i_2} \frac{1}{R_{i_1} R_{i_2}}\right)$$

$$+ \ldots + (-1)^{n-1} \frac{1}{R}\left(\sum_{i_1 < \ldots < i_{n-1}} \frac{1}{R_{i_1} \ldots R_{i_{n-1}}}\right) + (-1)^n \frac{1}{R_1 \ldots R_n} \Big\} = 0 \tag{II.2}$$

where $g_{\alpha\beta}$'s are the coefficients of the first fundamental form of the hypersurface M. The principal directions always exist and we can find an orthonormal system of principal directions.

Now let θ_α denote the angles between the direction v and the principal directions, where α runs from 1 to n. If we denote the principal directions by $t_1, \ldots, t_n$, then $\theta_1 \measuredangle (t_1, v), \ldots, \theta_n = \measuredangle (t_n, v)$.

The curvature $\frac{1}{R}$ in this direction v can be expressed in terms of the principal curvatures $\frac{1}{R_i}$, $i = 1, \ldots, n$, by means of Euler's formula

$$\frac{1}{R} = \sum_{i=1}^{n} \frac{1}{R_i} \sin^2\theta_i. \tag{II.3}$$

Now let us define a kind of normal curvature which we will denote by $\overline{R}$ and will be thought as a dual corresponding of R. This will be defined at the image point of m under the normal projection in the direction v of M. This concept has been defined by A. Mannheim (see [1] and [4]). From this dual viewpoint the Euler formula may be constructed as

$$\overline{R} = \sum_{i=1}^{n} R^*_i \sin^2\theta_i \tag{II.4}$$

where R^*_i shows the dual principal curvature corresponding to $\frac{1}{R_i}$.

Denoting by v_i the rectangular components of the unit vector v we write that

$$v = \sum_{i=1}^{n} v_i e_i. \tag{II.5}$$

Also we have that $\cos\theta_i = \langle v, e_i \rangle$, $i = 1, \ldots, n$. On multiplying both members of (II.5) by e_k we find that $v_k = \langle v, e_k \rangle$ and that $v_i = \cos\theta_i$, $i = 1, \ldots, n$. Consequently we have

$$v = \sum_{i=1}^{n} e_i \cos\theta_i \text{ or } \sum_{i=1}^{n} \cos^2\theta_i = 1. \tag{II.2}$$

III. ANGLES BETWEEN HYPERPLANES IN R^{n+1}

Let us consider two n-dimensional tangent vectors T_1^n, T_2^n which are n-planes in euclidean space R^{n+1} and t_1 and t_2 be the tangent vectors of the normal sections of T^n_1 and T^n_2 with hypersurface M. Also define the angles between the vectors t_1, t_2 and any vector in tangent space $T_M(m)$. To find this angles we will follow the procedure which has been given by H. Gluck [2]. The angle between a pair of lines in euclidean space R^{n+1} is the smaller of the two possible angles between any vectors parallel to these lines. The angle between a line and a hyperplane (that will be consider as a tangent vector to M) is the smallest angle between this line and any line in hyperplane. This is the same as the angle between a line and its orthogonal projection in hyperplane, or $\pi / 2$ in case this orthogonal projection degenerates to a point. Let us consider now a pair of hyperplanes of n–dimensional T_1^n and T_2^n in R^{n+1}. Suppose that among all pairs of lines, one from T_1^n and one from T_2^n, the lines t_1 and t_2 make the smallest possible angle, w_1, with each other. Let T_1^{n-1} and T_2^{n-1} be the orthogonal complements of t_1 and t_2 in T_1^n and T_2^n, respectively. Then it is easily seen that t_1 is orthogonal not only to T_1^{n-1} but also to T_2^{n-1}, and similarly t_2 is orthogonal not olny to T_2^{n-1} but also to T_1^{n-1}. If we iterate this procedure with T_1^{n-1} and T_2^{n-1} in the roles of T_1^n and T_2^n, we get another angle $w_2 = w_1$. Doing this n–times we get n angles $0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq \pi / 2$. This angles depend only on T_1^n and T_2^n, and not on the various choices possible during the above procedure, and these angle are called the principal angles between the hyperplane T_1^n and T_2^n. If we choose two orthonormal bases $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ for the subspaces V_1^n and V_2^n parallel to T_1^n and T_2^n such that $<u_i, v_i> = \cos w_i$ for $1 \leq i \leq n$ and $<u_i, v_j> = 0$ for $i \neq j$. Note that the orthogonal projection of v_i into V_2^n is $(\cos w_i)\, v_i$ and the orthogonal projection of v_i into V_i^n is $(\cos w_i) u_i$. Suppose that it is desired to find a single angle which might reasonably be called the angle between T_1^n and T_2^n. If one is forced to choose from among the principal angles, one would have to select the largest principal angle w_n for such a role, in order to insure that T_1^n and T_2^n are parallel if and only if the angle between them is zero. To arrive at the right definition carefully consider the case $n = 1$.

Then there is just one principal angle w_1 between the lines t_1 and t_2 and it coincides with the ordinary angle θ between these lines. This angle θ, lying between 0 and $\pi / 2$, has the following property. If U is any measurable subset of t_1 with one–dimensional measure s(U), then the orthogonal projection of U into t_2 is also measurable and has one–dimensional measure $(\cos\theta)$ s (U) in t_2. Similarly, if U′ is a measurable subset of t_2 with measure s(U′), then the orthogonal projection of U′ into t_1 has measure $(\cos\theta)$ s (U′) in t_1. Thus the angle θ between t_1 and t_2 may be defined as that angle between 0 and $\pi / 2$ whose cosine is the reduction factor for one–dimensional measure under orthogonal projection of t_1 into t_2, then (1x1) matrix of the orthogonal projection of $V^1{}_1$ into $V^1{}_2$ has a determinant whose absolute value is $\cos\theta$. So we can give the following definition directly.

Definition III.1. Let T_1^n and T_2^n be hyperplanes in R^{n+1}. Let the number p, $0 \leq p \leq 1$, be the reduction factor for n–dimensional measure under orthogonal projection of T_1^n into T_2^n. Then the unique angle θ, $0 \leq \theta \leq \pi/2$ such that $\cos\theta = p$, will be called the angle between T_1^n and T_2^n. The following theorem gives us the relation between the angle θ and the principal angles $w_1, \ldots, w_n$.

Theorem III.1. Let T_1^n and T_2^n be hyperplanes in R^{n+1}, and let $w_1 \leq w_2 \leq \ldots \leq w_n$ be the principal angles between them. Then the angle θ between T_1^n and T_2^n is given by $\cos\theta = \cos w_1 \ldots \cos w_n$. For a proof of this theorem see the paper [2]. To give a practical technique for computing the angle between two hyperplanes we will express the following theorem:

Theorem III.2. Let T_1^n and T_2^n be hyperplanes in R^{n+1}, and let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ be arbitrary bases for V_1^n and V_2^n, respectively. Then the angle θ between the hyperplanes is given by the formula

$$\cos\theta = \frac{|\det(u_i . v_j)|}{\sqrt{\det(u_i . u_j)}\sqrt{\det(v_i . v_j)}} . \qquad \text{(III.1.)}$$

This formula is the generalisation of the formula known for one dimensional two vectors in a vector space. Now we will give a classical concept which is called Dupin indicatrix.

Definition III.2. Dupin indicatrix I_m at each point m in M is the subset of $T_M(m)$ consisting of all vectors z such that $<Sz, z> = \pm 1$ and $Sz = \bar{D}_z N$, where S is the weingarten map and $\bar{D}$ is the natural connection defined on R^{n+1}, [3]. Now let $t_1, \ldots, t_n$ be an orthonormal set of eigen vectors of the map S^* which will be assumed as dual corresponding of the weingarten map S. Then $z = \sum_{j=1}^{n} a^j t_j$ and we write

$$<S^*z, z> = <\sum_{j=1}^{n} a^j S^* t_j, \sum_{j=1}^{n} a^j t_j)> = \sum_{j=1}^{n} (a^j)^2 <S^* t_j, t_j>$$

$$= \sum_{j=1}^{n} (a^j)^2 \frac{1}{R^*_1 \ldots \hat{R}^*_j \ldots R^*_n} \qquad \text{(III.2)}$$

where $\hat{R}^*_j$ indicates that the R^*_j is omitted as an argument.

Now let Q be a point in the intersection of I_m and T^n_o, then we illustrate the following figure in dimension 2.

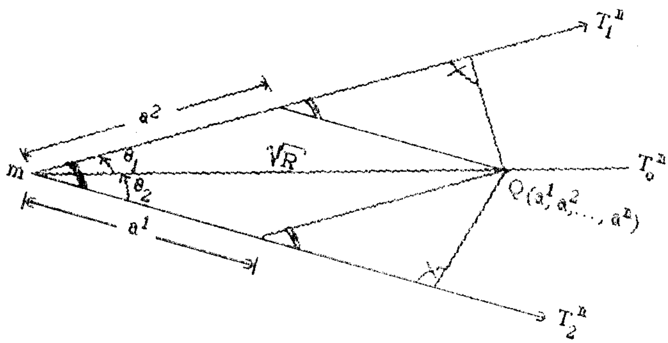

Figure III. 1.

By using figure III.1. for n–dimension we might infer that

$$a^j = \frac{\sqrt{R}\ \sin\ \theta_j}{\sin\ (\sum\limits_j\ \theta_j)}\ ,\ 1 \leq j \leq n, \qquad \text{(III.3.)}$$

where θ_j's are defined as in the Theorem III.1. Putting (III.3) into (III.2) we get from (II.4) that

$$\frac{\sin^2 (\Sigma\ \theta_j)}{R} = \frac{\sin^2 \theta_1}{R^*_1\ R^*_2\ \ldots\ R^*_n} + \frac{\sin^2 \theta_2}{R^*_1\ R^*_2\ \ldots\ R^*_n} + \cdots$$

$$+ \frac{\sin^2 \theta_n}{R^*_1\ \ldots\ R^*_{n-1}\ R^*_n} \qquad \text{(III.4)}$$

CONCLUSION

For the special case $\sum\limits_j \theta_j = \pi / 2$ we have $R_i = R_i^*$, $1 \leq i \leq n$, so the expression (III.4) changes into (II.3), but there is a slight difference that we will omit it here. (III.4) gives us a dual form of generalised Euler formula. Thus we get a relation between R and $\overline{R}$ by using (III.4) and (II.4). To get this we will use that

$$R^*_1\ \ldots\ \hat{R}^*_j\ \ldots\ R^*_n = \frac{1}{K\ R_j^*}\ ,$$ (K is the Gaussian curvature), so

we have that

$$R\ \bar{R} = \frac{\sin^2(\sum_j \theta_j)\,(R_1^*\sin^2\theta_1 + \dots + R^*_n \sin^2\theta_n)}{\dfrac{\sin^2\theta_1}{\dfrac{1}{K\ R^*_1}} + \dots + \dfrac{\sin^2\theta_n}{\dfrac{1}{K\ R^*_n}}}$$

or $K = \dfrac{\sin^2(\sum_j \theta_j)}{R\ \bar{R}}$. And finally for the special case $\sum_j \theta_j = \pi/2$ we find that

$$K = \frac{1}{R\ \bar{R}} \ . \qquad \text{(III.5)}$$

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