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FOURIER EXPANSIONS OF ENTIRE FUNCTIONS

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ABSTRACT

Let E denote a compact Jordan region in the complex plane with positive transfinite diameter and H(E) the Hilbert space of entire functions. This paper is deals with relations connecting the (p, q)-growth of an entire function $f \in H(E)$ with its Fourier coefficients with respect to an orthonormal sequence of polynomials in H(E). The (p, q)-order and generalized (p, q)-type have been characterized in terms of Fourier coefficients for $f \in H(E)$.

1. INTRODUCTION

Let Γ denote the class of all entire functions in the complex plane. The growth of an $f \in \Gamma$ is studied in terms of its (p, q)-order $\rho(p, q)$ and lower (p, q)-order $\lambda(p, q)$ and if $(b < \rho(p, q) < \infty)$ in terms of its (p, q)-type $T(p, q)$ also, such that

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r} = \begin{matrix}\rho(p, q)\\ \lambda(p, q)\end{matrix}. \tag{1.1}$$

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M(r, f)}{\left(\log^{[q-1]}\right)^{\rho(p,q)}} = T(p, q),\ o \le T(p, q) \le \infty, \tag{1.2}$$

where $b = 1$ if $p = q$, $b = 0$ if $p > q$, $M(r) \equiv M(r, f) = \max_{|z| = r} = |f(z)|$.

Since f(z) is entire, it can be represented by power series which converges uniformly to f(z) on every compact subset of the complex plane. Thus

$$f(z) = \sum_{n=0}^{\infty} a_n z^n , |z| < \infty . \tag{1.3}$$

We shall use the following notations throughout the paper:

$$\exp^{[m]}x = \log^{[-m]}x = \exp(\exp^{[m-1]}x) = \log(\log^{[-m-1]}x), \ m = \pm 1, \pm 2, \ldots$$

$$\Delta_{[s]}(x) = \prod_{i=0}^{r} \log^{[i]}x \ \text{for} \ r = 0, 1, \ldots$$

$$p(L(p, q)) = \begin{cases} L(p, q) & \text{if } q < p < \infty \\ 1 + L(p, q) & \text{if } p = q = 2 \\ \max(1, L(p, q)) & \text{if } 3 \le p = q \\ \infty & \text{if } p = q = \infty \end{cases}$$

The characterisations of above growth constants in terms of coefficients a_n's are known ([2], [3]). Thus

$$\rho(p, q) = P(L(p, q)) \tag{1.4}$$

where

$$L(p, q) = \limsup_{k\to\infty} \frac{\log^{[p-1]}n}{\left(\log^{[q]}\right) |a_n|^{-1/n}}$$

$$\frac{T(p, q)}{M} = \limsup_{k\to\infty} \frac{\log^{[p-2]}n}{\left(\log^{[q-1]} |a_n|^{-1/n}\right)^{\rho-A}} \tag{1.5}$$

where

$$M = \begin{cases} (\rho(2, 2)-1)^{\rho(2, 2)-1}/(\rho(2, 2))^{\rho(2, 2)} & \text{if } (p, q) = (2, 2), \\ 1/e\rho\ (2, 1) & \text{if } (p, q) = (2, 1), \\ 1 & \text{otherwise.} \end{cases} \tag{1.6}$$

and $A = 1$ if $(p, q) = (2, 2)$, $A = 0$ if $(p, q) \ne (2, 2)$.

Let Δ stand for the closed unit disc i.e., $\{z: |z| \le 1\}$, and let $H(\Delta)$ denote the Hilbert space of functions holomorphic in Δ with inner product

$$(f, g) = \iint_\Delta f(z)\bar{g}(z)ds \quad , \quad f, g \in H(\Delta) \tag{1.7}$$

It is well known (see [1]) that the sequence $A(\Delta) \equiv \left\{\frac{\sqrt{n}}{\pi} z^{n+1}\right\}_{n=1}^{\infty}$ forms a complete orthonormal sequence in $H(\Delta)$ and that the Fourier coefficients b_n ($n = 1, 2, 3, \ldots$) of $f \in \Gamma \subset H(\Delta)$, defined (1.3) are given by

$$b_n = \frac{\sqrt{\pi}}{n} a_{n-1} \quad , \quad n = 1, 2, \ldots \tag{1.8}$$

It is easily seen that the relations (1.4) and (1.5) continue to hold if in these relations we replace a_n by b_n. Thus (1.4) and (1.5) may be interpreted as describing the growth of $f \in \Gamma$ in terms of its Fourier coefficients with respect to set $A(\Delta)$.

Let E be a closed and bounded Jordan region in complex plane with transfinite diameter $d > 0$ and $H(E)$ the Hilbert space of functions holomorphic in E with inner product

$$(f, g) = \iint_E \omega(z) f(z) \overline{g}(z) ds, \quad f, g \in H(E) \tag{1.9}$$

where $\omega(z)$ is a positive, continuous function on E. If $A(E) \equiv \{p_{n-1}(z)\}_{n-1}^{\infty}$, $p_n(z)$ being a polynomial of degree not exceeding n, is a complete orthonormal sequence of $H(E)$, a question that arises naturally is, "Do the relation (1.4) and (1.5) continue to hold if a_n is replaced by the Fourier coefficient of $f \in \Gamma \subset H(E)$ with respect to the set $H(E)$? Rizvi and Juneja [7] have answered this question in affirmative except that a factor $d^{-\rho}$ appears in the right hand side of (1.5) for $(p, q) = (2, 1)$. These results obviously leave a big class of entire functions, such as entire functions of slow growth or of fast growth etc. It has been noticed that these authors fail to compare the above coefficients of those entire functions which have same positive finite order but their types are infinity. For the view point of including this important class of entire functions, we shall utilise the concept of proximate order due to Levin [5]. Moreover, for including entire function of slow growth and fast growth their results will also be extended to (p, q)-scale introduced by Juneja et al. ([2], [3]).

2. DEFINITIONS AND AUXILLIARY RESULTS

Definition 1. A positive function $\rho_{p,q}(r)$ defined on $\{[r_0, \infty), r_0 > \exp^{[q-1]}1\}$ is said to be a proximate order of an entire function with index pair (p, q) if

(i) $\rho_{p,q}(r) \to \rho(p, q)$ as $r \to \infty$, $b < \rho(p, q) < \infty$,

(ii) $\Delta_{[q]}(r)\, \rho_{p,q}(r) \to 0$ as $r \to \infty$; $\rho'_{p,q}(r)$ denotes the derivative of $\rho_{p,q}(r)$.

It is known [6, Thm. 4] that $(\log^{[q-1]} r)^{\rho_{p,q}(r)-A}$ is a monotonically increasing function of r for $r > r_0$.

Hence we can define the function $\phi(x)$ for $x > x_0$ to be the unique solutions of the equation,

$$x = (\log^{[q-1]} r)^{\rho_{p,q}(r)-A} \Leftrightarrow \phi(x) = \log^{[q-1]} r.$$

Definition 2. Let $f(z)$ be an entire function of (p, q)-order $\rho(p, q)$ $(b < \rho(p, q) < \infty)$ such that

$$\limsup_{r \to \infty} \frac{\log^{[p-1]} M(r, f)}{\left(\log^{[q-1]} r\right)^{\rho_{p,q}(r)}} = T^*(p, q), \quad 0 \le T^*(p, q) \le \infty .$$

If the quantity $T^*(p,q)$ is different from zero and infinite then $\rho_{p,q}(r)$ is said to be the proximate order of a given entire function $f(z)$ and $T^*(p, q)$ as its generalized (p, q)-type.

Definition 3. An entire function with index-pair (p, q) is said to be of minimal, normal and maximal (p, q)-type with respect to a proximate order according as $T^*(p, q)$ as zero, positive finite and infinite respectively.

Let E_r be the curve defined by

$$E_r = \{z: |\psi(z)| = r\},\ r > 1,$$

where $\omega = \psi(z)$ is the univalent holomorphic function mapping the complement of E onto $|\omega| > 1$ such that $\psi(\infty) = \infty$, $\psi'(\infty) = 1$ and let D_r be the domain interior to E_r. Also, set $\overline{M}(r, f) = \sup_{z \in E_r} |f(z)|$ for $r > 1$.

The following auxilliary results will be utilized in the sequal.

Lemma 1. If $f(z)$ is an entire function of (p, q)-order $\rho(p, q)$ then

$$\limsup_{r \to \infty} \frac{\log^{[p]} \overline{M}(r, f)}{\log^{[q]} r} = \rho(p, q) ,$$

and for $\rho(p, q)$ $(b < \rho(p, q) < \infty)$, $T^*(p, q)$ is given by

$$\limsup_{r \to \infty} \frac{\log^{[p-1]} \overline{M}(r, f)}{\left(\log^{[q-1]} r\right)^{\rho_{p,\, q}(r)}} = \frac{T^*(p, q)}{\beta} .$$

Proof. This follows easily from Winiarski [8].

Lemma 2. $f(z) \in H(E)$ is an entire function if and only if

$$\lim_{n \to \infty} |C_n|^{1/n} = 0 ,$$

$$C_n = \iint_E \omega(z) f(z) \overline{p_n}(z) ds,\ p_n(z) \in A(E),\ n = 0, 1, 2, \ldots \tag{2.1}$$

Lemma 3. Let $f \in H(E)$. Given $\varepsilon > 0$, there exists an integer N depending on ε and E such that

$$|C_{n+1}| < K \frac{\overline{M}(r, f)}{r^n} e^{n\varepsilon}$$

for all r such that $R > r \geq r_0(\varepsilon)$ and $n \geq N$, where, $\overline{M}(r) = \max_{z \in E_r} |f(z)|$,C_n are defined by (2.1) and K is independent of n and r.

Proof of Lemmas 2 and 3 follows from [7].

3. MAIN RESULTS

In this section we obtain expressions for the (p, q)-order $\rho(p, q)$ and generalized (p, q)-type of an entire function $f \in H(E)$ in terms of its Fourier coefficients C_n. Thus, we have

Theorem 1. If $f \in H(E)$ is an entire function of (p, q)-order $\rho(p, q)$ $(b < \rho(p, q) < \infty)$ and generalized (p, q)-type $T^*(p, q)$, then for every C_n, there exist an entire function $g(z) = \sum_{n=0}^{\infty} |C_n| z^n$, such that

$$\rho(p, q, f) = \rho(p, q, g) \text{ and } T^*(p, q, f = \beta T^*(p, q, g) \tag{3.1.}$$

where

$$\beta = d^{-\rho(p,\, 1)} \text{ for } q = 1 \text{ and } \beta = 1 \text{ for } q > 1.$$

Proof. We observe that by ([7], Lemma 3)

$$|f(z)| \leq M_0 \sum_{n=0}^{\infty} |C_n| r^n (1 \to \varepsilon)^n \text{ for } z \text{ in } E_r ,$$

M_0 being a constant. So

$$\overline{M}(r) \leq M_0 g(r(1+\varepsilon)) = M_0 M(r(1+\varepsilon);\ g) . \tag{3.2}$$

Using Lemma 1 in (3.2) we obtain

$$\rho(p,\ q,\ f) \leq \rho(p,\ q,\ g) \text{ and } T^*(p,\ q,\ f) \leq \beta T^*(p,\ q,\ g). \tag{3.3}$$

In view of Lemma 2, g(z) is an entire function. By Lemma 3 we have for any $\varepsilon > 0$,

$$|C_{n+1}| < K \frac{\overline{M}\ (r,\ f)}{r^n} e^{n\varepsilon} \tag{3.4}$$

Using (3.4) in the power series expansion of g(z), we get

$$g\left(r/de^{2\varepsilon}\right) = \sum_{n=0}^{\infty} |C_n| \left(r/de^{2\varepsilon}\right)^n$$

$$\leq \frac{Kr\overline{M}(r,\ f)}{de^{2\varepsilon}} \sum_{n=0}^{\infty} 1/e^{n\varepsilon}$$

$$\leq \frac{Kr\overline{M}(r,\ f)}{de^{2\varepsilon}\left(e^{\varepsilon-1}\right)} ,$$

or

$$\log g\left(r/de^{2\varepsilon}\right) \leq O(1) + \log \overline{M}(r,\ f) + \log r .$$

Thus in view of above inequality and Lemma 1, for $p \geq 2$ and $q=1$,

$$\rho(p,\ 1,\ q) \leq \rho(p,\ 1,\ f) \text{ and} \tag{3.5}$$

$$T^*(p,\ 1,\ g) \leq e^{2\varepsilon\rho(p,\ 1)}\ d^{\rho(p,\ 1)}\ T^*(p,\ 1,\ f)$$

and for $p \geq 2$, and $q > 1$,

$$\rho(p,\ q,\ g) \leq \rho(p,\ q,\ f) \text{ and } T^*(p,\ q,\ g) \leq T^*(p,\ q,\ f). \tag{3.6}$$

Since $\varepsilon < 0$ is arbitrary, both inequalities imply that for all (p, q),

$$\rho(p,\ q,\ g) \leq \rho(p,\ q,\ f) \text{ and } \beta T^*(p,\ q,\ g) \leq T^*(p,\ q,\ f) . \tag{3.7}$$

Combining (3.3) and (3.7) we get the required results.

Theorem 2. $f \in H(E)$ is an entire function of (p, q)-order $\rho(p, q)$ $(b < \rho(p, q) < \infty)$ if and only if

$$\rho(p, q) = \rho(L(p, q)),$$

where

$$L(p, q) = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q-1]} |C_n|^{-1/n}}.$$

Proof. By Lemma 2, we have concluded that $f \in H(E)$ is an entire function if and only if g(z) is an entire function. Moreover by Theorem 1, f(z) and g(z) have same (p, q)-order. Applying Corollary 1 by Juneja et al. [2, p.62] to the function

$$g(z) = \sum_{n=0}^{\infty} |C_n| z^n$$ Theorem 2 follows.

Remark 1. For (p, q) = (2, 1), this theorem includes Theorem 1 by Rizvi and Juneja [7].

Theorem 3. If $f(z) \in H(E)$ is an entire function of (p, q)-order $\rho(p, q)$ $(b < \rho(p, q) < \infty)$ and generalized (p, q)-type $T^*(p, q)$ $(0 < T^*(p, q) < \infty)$ if and only if

$$\frac{T^*(p, q)}{\beta M(p, q)} = \limsup_{n \to \infty} \left[\frac{\phi\left(\log^{[p-2]} n\right)}{\log^{[q-1]} |C_n|^{-1/n}}\right]^{\rho(p, q)-A}.$$

Proof. To prove this theorem we apply Theorem 1 by Kasana [4] to the function g(z) and resulting charaterisation of $T^*(p, q, g)$ in terms of C_n and the relation $T^*(p, q, f) = \beta T^*(p, q, g)$ taking together prove the theorem.

Taking $\rho_{p,q}(r) = \rho(p, q)$ $\forall\ r > r_0$ and $\phi(x) = x^{1/\rho(p, q)-A}$, we have the following corollary which gives a formula for (p, q)-type T(p, q) in terms of Fourier coefficients of an entire function f(z).

Corollary 3. $f(z) \in H(E)$ is an entire function having (p, q)-order $\rho(p, q)$ $(b < \rho(p, q) < \infty)$ and (p, q)-type $T(p, q)$ $(0 < T(p, q) < \infty)$ if and only if

$$\frac{T(p, q)}{\beta M(p, q)} = \limsup_{n \to \infty} \frac{\log^{[p-2]} n}{\left(\log^{[q-1]} |C_n|^{-1/n}\right)^{\rho(p, q)-A}} .$$

Remark 2. For (p, 2) = (2, 1) this corollary gives the Theorem 2 of Rizvi and Juneja [7] as a particular case.

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