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A NOTE ON THE FRACTAL DIMENSION

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ABSTRACT

We give a generalization of a theorem from Barnsley [1] which simplifies the computation of the fractal dimension.

1. INTRODUCTION

Mandelbrot has defined a fractal to be a set with Hausdorff dimension strictly greater than its topological dimension. There are many other definitions of dimensions, although the Hausdorff dimension is commonly used. Tricot did a study of 12 definitions of dimension. Most definitions have some restrictions on the ε–covers considered in the definitions of measure. In some situations these definitions are more natural for the applications. Sometimes it is just too hard to find the Hausdorff dimension, but calculation is possible for other definitions. One of them is the fractal dimension (box dimension).

Definition 1.1. Let (X, d) be a metric space and $A \subset X$. For each $\varepsilon > 0$, let $N(A, \varepsilon)$ denote the minimal number of closed balls of radius $\varepsilon > 0$ needed to cover A.

If

$$D = \lim_{\varepsilon \to 0} \frac{\log (N(A, \varepsilon))}{\log \frac{1}{\varepsilon}}$$

exists, then D is called fractal dimension of A.

The following theorem from Barnsley [1] simplifies the process of calculating the fractal dimension. It allows one to replace the continuous variable ε by a discrete variable.

Theorem 1.1. Let (X, d) be a metric space and $A \subset X$ a compact subset. Let $\varepsilon_n = Cr^n$ for real numbers $0 < r < 1$ and $C > 0$, and integers $n = 1, 2, \ldots$ If

$$D = \lim_{n \to \infty} \frac{\log (N(A, \varepsilon_n))}{\log \dfrac{1}{\varepsilon_n}}$$

then A has fractal dimension D [1].

We now give a slight generalization of this theorem for $A \subset R^n$

Theorem 1.2. Let $A \subset R^m$ be a compact subset, $\varepsilon > 0$ and $N(\varepsilon)$ the smallest number of closed balls of radius $\varepsilon > 0$ needed to cover A. Let $r_n = Cr^n$, $0 < r < 1$, $n \in N$, $C \in R^+$ and ε_n is a null–sequence of positive numbers such that for every $n \in N$, $n \geq 1$ there exist positive numbers c_1 and c_2 satisfying $c_1 < \dfrac{r_n}{\varepsilon_n} < c_2$. If

$$D = \lim_{\varepsilon_n \to 0} \frac{\log (N(\varepsilon_n))}{\log \dfrac{1}{\varepsilon_n}}$$

exists then A has fractal dimension D.

Proof: It can be shown that

1. If $\delta < \varepsilon$ then $N(\varepsilon) \leq N(\delta)$

2. If $A \subset R^m$, $c \in R^+$ then $\left(\dfrac{1}{c+1}\right)^m N(\varepsilon) \leq N(c\varepsilon) \leq \left(\dfrac{1}{c} + 1\right)^m N(\varepsilon)$

By the hypotesis and from 1, it follows that

$$N(c_2\varepsilon_n) \leq N(r_n) \leq N(c_1\varepsilon_n)$$

and by using 2 one obtains

$$\left(\frac{1}{c_2+1}\right)^m N(\varepsilon_n) \leq N(c_2\varepsilon_n) \leq N(r_n) \leq N(c_1\varepsilon_n) \leq \left(\frac{1}{c_1} + 1\right)^m N(\varepsilon_n)$$

$$\left(\frac{1}{c_2+1}\right)^m N(\varepsilon_n) \leq N(r_n) \leq \left(\frac{1}{c_1} + 1\right)^m N(\varepsilon_n)$$

$$\log\left(\frac{1}{c_2+1}\right)^m N(\varepsilon_n) \leq \log N(r_n) \leq \log\left(\frac{1}{c_1}+1\right)^m N(\varepsilon_n)$$

$$\lim_{n\to\infty} \frac{\log\left(\frac{1}{c_2+1}\right)^m N(\varepsilon_n)}{\log\frac{1}{\varepsilon_n}} \leq \lim_{n\to\infty} \frac{\log N(r_n)}{\log\frac{1}{\varepsilon_n}} \leq \lim_{n\to\infty} \frac{\log\left(\frac{1}{c_2}\right)^m N(\varepsilon_n)}{\log\frac{1}{\varepsilon_n}}$$

$(\varepsilon_n < 1)$

$$\lim_{n\to\infty} \frac{-m\log(c_2+1)+\log(N(\varepsilon_n))}{-\log\varepsilon_n} \leq \lim_{n\to\infty} \frac{\log(N(r_n))\log r_n}{-\log r_n \log\varepsilon_n} \leq$$

$$\lim_{n\to\infty} \frac{m\log\left(\frac{1}{c_1}+1\right)+\log(N(\varepsilon_n))}{-\log\varepsilon_n}$$

Since $\lim_{n\to\infty} \frac{\log r_n}{\log\varepsilon_n} = 1$, we obtain

$$\lim_{n\to\infty} \frac{\log(N(\varepsilon_n))}{\log\frac{1}{\varepsilon_n}} = \lim_{n\to\infty} \frac{\log(N(r_n))}{\log\frac{1}{r_n}}$$

Hence, by Theorem 1.1, the fractal dimension of A exists and equals to D.

REFERENCES

[1] BARNSLEY, M.F., Fractals Everywhere, Academic Press, 1988 Orlando, FL.

[2] TRICOT, C., Douze definitions de la densité logarithmique, C.R. Acad. Sci. Paris Sér. I Math. 293 (1981), 549–552.