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POLYNOMIAL SOLUTIONS FOR A CLASS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, polynomial solutions are given for a class of linear non-homogeneous singular partial differential equations of the second order. At the end of this paper, polynomial solutions are given for an iterated equation with order 2p which is obtained by applying the operator belonging to the same equation consecutively.

INTRODUCTION

Consider the following linear nonhomogeneous singular partial differential equation,

$$Lu = \frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + \frac{\alpha}{x}\frac{\partial u}{\partial x} + \frac{\beta}{y}\frac{\partial u}{\partial y} = q(x,y) \ . \tag{1}$$

where b, α and β are any real constants and q is a polynomial in R^2. Clearly the equation (1) includes some of the well-known classical equations such as the Laplace equation, the Poisson equation, the axially symmetric potential equation and the wave equation. To obtain a particular solution for the equation (1) in the case of $q(x,y) \neq 0$ is an important problem. From the theory of linear equations it is known that if we have a particular solution of the equation Lu=q and we know the general solution of the equation Lu=0, then we can obtain the general solution of the equation Lu=q. In the equation (1), if $g(x,y) \equiv 0$ the polynomial solutions are given in [2,4]. In this paper, polynomial solutions are given for the equation (1) and polynomial solutions are given for the iterated equation $L^p(u) = 0$ for $p \geq 1$. The iterated operator L^p is defined by the relation.

$$L^{s+1}(u) = L[L^s(u)] \qquad s = 1, \dots, p-1$$

2. POLYNOMIAL SOLUTIONS FOR THE EQUATION (1)

In generaly a polynomial q(x,y) may be written in the form

$$q(x,y) = \sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij}x^i y^j \ ; \ M,N \in N. \tag{2}$$

If $q_{ij} = x^i y^j$ $0 \le i \le M$, $0 \le j \le N$, then we can write $q(x,y) = \sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij}q_{ij}$. By the principle of superposition, it is known that if $Lp_{ij} = q_{ij}$, then we obtain

$$L\left(\sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij}p_{ij}\right) = \sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij}L(p_{ij}) = q(x,y) \ .$$

Hence, it is clear that for obtaining a particular solution p of Lu = q, it will be enough to find particular solutions $u = p_{ij}$ satisfying

$$Lu = x^i y^j \qquad i,j \in N \tag{3}$$

Thus, $\sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij}p_{ij} = p$ becomes the required particular solution of the equation Lu=q. We explain below how the polynomial solutions are obtained when the typical terms on the right-hand side of the equation (3) are of the form $x^i y^{2j}$, $x^{2i} y^j$, $x^{2i} y^{2j}$.

Theorem 1. Let $0 \le i \le M$, $0 \le j \le N$; M, N $\in$ N be nonnegative integers. Then the equation

$$Lu = \frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + \frac{\alpha}{x}\frac{\partial u}{\partial x} + \frac{\beta}{y}\frac{\partial u}{\partial y} = x^i y^{2j} \tag{4}$$

has a polynomial solution

$$p = \frac{1}{(i+2)[(i+1)+\alpha]}x^{i+2}y^{2j} + \sum_{s=2}^{j+1} b_{2s}x^{i+2s}y^{2j-2s+2} \tag{5}$$

where

$$b_{2s} = (-1)^{s-1}\frac{2j(2j-2)\dots(2j-2s+4)[(2j-1)b+\beta]\dots[(2j-2s+3)b+\beta]}{(i+2)\dots(i+2s)[(i+1)+\alpha]\dots[(i+2s-1)+\alpha]} \tag{6}$$

$s = 2, \dots, j+1$ and for $s = 1, \dots, j+1$, $\alpha \ne -(i+2s-1)$.

Proof. Because of the property of the operator L, a particular solution p(x,y) of the equation (4) should be a polynomial consisting of the terms of degree (i+2j+2). Thus p(x,y) can be chosen as

$$\begin{aligned} p = {} & b_2 x^{i+2} y^{2j} + b_4 x^{i+4} y^{2j-2} + b_6 x^{i+6} y^{2j-4} + \dots \\ & + b_{2s-2} x^{i+2s-2} y^{2j-2s+4} + b_{2s} x^{i+2s} y^{2j-2s+2}\ ;\ 2j-2s+2 \geq 0 \end{aligned} \tag{7}$$

Now we calculated Lp from (4) and (7)

$$\begin{aligned} Lp = {} & (i+2)[(i+1)+\alpha]b_2 x^i y^{2j} + \{(i+4)[(i+3)+\alpha]b_4 + 2j[(2j-1)b+\beta]b_2]\} x^{i+2} y^{2j-2} \\ & + \{(i+6)[(i+5)+\alpha]b_6 + (2j-2)[(2j-3)b+\beta]b_4\} x^{i+4} y^{2j-4} + \dots \\ & + \{(i+2s)[i+2s-1)+\alpha]b_{2s} + (2j-2s+4)[2j-2s+3)b+\beta]b_{2s-2}\} x^{i+2s-2} y^{2j-2s+2} \\ & + (2j-2s+2)[(2j-2s+1)b+\beta]b_{2s} x^{i+2s} y^{2j-2s} = x^i y^{2j}. \end{aligned}$$

Equating the coefficients of similar terms on both sides of the above identity, the following relations;

$$b_2 = \frac{1}{(i+2)[(i+1)+\alpha]}$$

is obtained from $(i+2)[(i+1)+\alpha]b_2 = 1$. Similarly, the other coefficients have the following forms.

$$\begin{aligned} b_4 &= -\frac{2j[(2j-1)b+\beta]}{(i+4)[(i+3)+\alpha]} b_2 , \\ b_6 &= -\frac{(2j-2)[(2j-3)b+\beta]}{(i+6)[(i+5)+\alpha]} b_4 , \\ b_{2s} &= -\frac{(2j-2s+4)[(2j-2s+3)b+\beta]}{(i+2s)[(i+2s-1)+\alpha]} b_{2s-2} . \end{aligned}$$

By multiplying them side by side and writing the value of b_2, we obtain b_{2s} as defined in (6). Hence, we obtain p(x,y) as given in (5).

Theorem 2. Let $0 \leq i \leq M$, $0 \leq j \leq N$; $M,N \in N$ be nonnegative integers. Then the equation

$$Lu = \frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + \frac{\alpha}{x}\frac{\partial u}{\partial x} + \frac{\beta}{y}\frac{\partial u}{\partial y} = x^{2i} y^j$$

has a polynomial solution

$$p = \frac{1}{(j+2)[(j+1)b+\beta]} x^{2i} y^{j+2} + \sum_{s=2}^{i+1} b_{2s} x^{2i-2s+2} y^{j+2s} \tag{8}$$

where

$$b_{2s} = (-1)^{s-1} \frac{2i(2i-2)\dots(2i-2s+4)[2i-1+\alpha]\dots[2i-2s+3+\alpha]}{(j+2)\dots(j+2s)[(j+1)b+\beta]\dots[(j+2s-1)b+\beta]}$$

$s = 2, \ldots, i+1$ and for $s = 1, \ldots, i+1$ $\beta \neq -(j+2s-1)b$.

The proof is very similar to the proof of Theorem 1; so we shall not give it here. On the other hand, if the right-hand side of the equation (3) is of the form $x^{2i}y^{2j}$, replacing i by 2i in (5) or j by 2j in (8), we simply obtain polynomial solutions of $Lu = x^{2i}y^{2j}$. Hence, if typical terms in q are of the forms $x^i y^{2j}$, $x^{2i}y^j$, $x^{2i}y^{2j}$, we obtain polynomial solution of the equation (1).

Special Cases. If q is a polynomial which is odd with respect to the variables x and y in its terms, then the equation (1) has polynomial solutions in some special cases. Namely, if the typical term is of the form $x^{2n+1}y^{2m+1}$ in q ($n,m \geq 1$), in the following special cases, we find polynomial solutions of the equation

$$Lu = x^{2n+1}y^{2m+1} \tag{9}$$

If we choose polynomial solution

$$p = Ax^{2n+3}y^{2m+1} + Bx^{2n+1}y^{2m+3} \tag{10}$$

for the equation (9) by applying the operator L in (9) to this function, we obtain

$$\begin{aligned} L(p) &= A(2m+1)[2mb+\beta]\}.x^{2n+3}y^{2m-1} \\ &+ A(2n+3)[2(n+1)+\alpha]+B(2m+3)[2(m+1)b+\beta]\}.x^{2n+1}y^{2m+1} \\ &+ B(2n+1)(2n+\alpha)x^{2n-1}y^{2m+1} \equiv x^{2n+1}y^{2m+1}. \end{aligned}$$

From this identity, we obtain

$$\begin{aligned} &A(2m+1)(2mb+\beta) = 0 \\ &B(2n+1)(2n+\alpha) = 0 \\ &A(2n+3)[2(n+1)+\alpha]+B(2m+3)[2(m+1)b+\beta] = 1 \end{aligned}$$

We can write the following special cases here.

i. The equation (9) has a polynomial solution

$$p = \frac{1}{(2m+3)[2(m+1)b+\beta]} x^{2n+1}y^{2m+3} \quad \text{for } \alpha = -2n$$

ii. The equation (9) has a polynomial solution

$$p = \frac{1}{(2n+3)[2(n+1)+\alpha]} x^{2n+3}y^{2m+1} \quad \text{for } \beta = -2mb$$

iii. If A and B are nonzero arbitrary constants which satisfy the equality $A(2n+3)[2(n+1)+\alpha]+B(2m+3)[2(m+1)b+\beta] = 1$, the equation (9) has a polynomial solution $p = Ax^{2n+3}y^{2m+1} + Bx^{2n+1}y^{2m+3}$ for $\alpha = -2n$, $\beta = -2mb$.

3. POLYNOMIAL SOLUTIONS FOR THE EQUATION $L^p(u) = 0$

The formula for the operator L is easily derived.

$$L(fg) = gL(f) + 2\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + b\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right) + fL(g) \tag{11}$$

In particular, if f is replaced by x^k ($k \in R$) in (11), we obtain

$$L(x^k g) = k(k-1+\alpha)x^{k-2}g + 2kx^{k-1}\frac{\partial g}{\partial x} + x^k L(g) \ . \tag{12}$$

Lemma 1. Let $T^* = x\frac{\partial}{\partial x}$ and if $Lg = 0$, then

$$L(x^k g) = x^{k-2}k(k-1+\alpha+2T^*)g \tag{13}$$

Proof. The proof is obvious from (12).

Lemma 2. Let $L_x = \frac{\partial^2}{\partial x^2} + \frac{\alpha}{x}\frac{\partial}{\partial x}$ and A_j, B_j, C_j, D_j be real constants, then the functions

$$u_j(x,y) = A_j x^{1-\alpha} + B_j y^{1-\frac{\beta}{b}} + C_j x^{1-\alpha} y^{1-\frac{\beta}{b}} + D_j \tag{14}$$

are solutions of both the equations $L(u_j) = 0$ and $L_x(u_j) = 0$.

Proof. By applying the operator L and L_x to this function u_j we simply see that $L(u_j) = 0$ and $L_x(u_j) = 0$.

Lemma 3. If g is of the form (14), then

$$L^p(x^k g) = x^{k-2p}\prod_{j=0}^{p-1}(k-2j)[k-1+\alpha+2T^*-2j]\, g \ . \tag{15}$$

Proof. We prove this by the method of induction. Let $T = k-1+\alpha+2T^*$. From (13), we write $L(x^k g) = kx^{k-2}(Tg)$. Applying the operator L on both sides of this equality and using (12), we obtain

$$L^2(x^k g) = kL[x^{k-2}Tg] = k\left\{x^{k-4}(k-2)[(k-2)-1+\alpha]Tg + 2(k-2)\ x^{k-3}\frac{\partial Tg}{\partial x}\right\} + kx^{k-2}L(Tg) \ . \tag{16}$$

On the other hand, by direct calculation, it can be shown that

$$LT^* = T^*L + 2L_x \tag{17}$$

$$LT = (k-1+\alpha)L + 2T^*L + 4L_x \tag{18}$$

Here, let L_x be the same as in Lemma 2. From (18), we have $L(Tg) = 0$. In (16), by making use of $L(Tg) = 0$, we obtain

$$L^2(x^k g) = k(k-2)x^{k-4}[(k-3)+\alpha+2T^*]Tg.$$

Now, first assume that the equality (15) is true for p and show that it is true for p+1. Applying the operator L on both sides of the equality (15) and using (12), we obtain

$$\begin{aligned} L^{p+1}(x^k g) &= L\left\{x^{k-2p}\prod_{j=0}^{p-1}(k-2j)[T-2j]\,g\right\} \\ &= (k-2p)[(k-2p-1)+\alpha]\,x^{k-2(p+1)}\prod_{j=0}^{p-1}(k-2j)[T-2j]\,g \\ &\quad + 2(k-2p)\,x^{k-2(p+1)}\,x\frac{\partial}{\partial x}\prod_{j=0}^{p-1}(k-2j)[T-2j]\,g \\ &\quad + x^{k-2p}L\left\{\prod_{j=0}^{p-1}(k-2j)[T-2j]\,g\right\} \end{aligned}$$

By making use of $L(Tg) = 0$ we see that

$$L\left\{\prod_{j=0}^{p-1}(k-2j)[T-2j]\,g\right\} = 0\ .$$

Hence, we obtain

$$L^{p+1}(x^k g) = x^{k-2(p+1)}\prod_{j=0}^{p}(k-2j)[k-1+\alpha+2T^*-2j]\,g\ .$$

This completes the proof.

If f is replaced by y^k ($k \in R$) in (11), we give the following Lemma. Its proof is very simiar to the proof of Lemma 3; so we shall give it here without proof.

Lemma 4. If g is of the form of (14), then

$$L^p(y^k g) = y^{k-2p}\prod_{j=0}^{p-1}(k-2j)[b(k-1-2j+2T^*)+\beta]\,g\ .$$

Theorem 3. If the functions g_j and h_j are of the form (14), then the polynomial solution of $L^p(u) = 0$ for $p \geq 1$ is given by

$$u = \sum_{j=0}^{p-1} (x^{2j} g_j + y^{2j} h_j)$$

where $1 > \alpha \in Z$ and $1 > \frac{\beta}{b} \in Z$.

Proof. It is known that $L^p(x^{2j} g_j) = 0$ $(j = 0, 1, ..., p-1)$ from Lemma 3, and $L^p(y^{2j} h_j) = 0$ from Lemma 4. Because of the linearity of L^p the function

$$u = \sum_{j=0}^{p-1} (x^{2j} g_j + y^{2j} h_j)$$

satisfies the equation $L^p(u) = 0$ where u is a polynomial for $1 > \alpha \in Z$ and $1 > \frac{\beta}{b} \in Z$.

REFERENCES

[1] ALTIN, A., Some expansion formulas for a class of singular partial differential equation, Proc. Amer. Math. Soc., 85 (1982), 42-46.

[2] MILES, E.P. JR., and WILLIAMS, E., A basic set of homogeneous harmonic polynomials in k variables, Proc. Amer. Math. Soc., Vol. 6 (1955), 191-194.

[3] MILES, E.P. JR., WILLIAMS, E., A basic set of polynomial solutions for the Euler-Poisson Darboux and Beltrami equations, Amer. Math. Monthly Vol. 63 (1956), 401-404.

[4] MILES, E.P. JR., YOUNG, E.C., Basic sets of polynomials for generalized Beltrami and Euler-Poisson Darboux equations and their iterates, Proc. Amer. Math. Soc., 18 (1967), 981-986.

[5] YEH, R.Z., Solutions of polyharmonic Dirichlet problems derived from general solutions in the plane. J. Math. Anal. Appl. 154 (1991), 341-363.

[6] YEH, R.Z., Polynomial solutions of poisson and polyharmonic equation in R^n Journal of Mathematical Analysis and Applications, Vol. 174 (1993), 467-475.