Commun. Fac. Sci. Univ. Ank. Series A1
V. 47. pp. 151-156 (1998)

ON APPLICATION OF THE YANO-AKO OPERATOR IN THE THEORY OF LIFTS

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(Received May 18, 1998; Revised Sep. 28, 1998; Accepted Nov. 5, 1998)

ABSTRACT

In this paper, it was obtained, by using the Yano-Ako operator, the complete lift of the tensor structure $S \in T^1_2(M_n)$ along the pure cross-section of $T^1_q(M_n)$ as: ${}^cS^i_{jk} = S^i_{jk}$, ${}^cS^{i}_{\bar{j}k} = {}^cS^{i}_{j\bar{k}} = {}^cS^{i}_{\bar{j}\bar{k}} = 0$, ${}^cS^{\bar{i}}_{jk} = -\Phi^{S}_{ij} t^{i_1}_{j_1 \dots j_q}$, ${}^cS^{\bar{j}}_{m\bar{i}} = S^{i_1}_{ml}\delta^{l_1}_{j_1} \dots \delta^{l_q}_{j_q}$, ${}^cS^{\bar{j}}_{\bar{i}\bar{k}} = 0$ and ${}^cS^{\bar{j}}_{\bar{l}m} = S^{i_1}_{lm}\delta^{l_1}_{j_1} \dots \delta^{l_q}_{j_q}$, where Φ^S is Yano-Ako operator.

1. INTRODUCTION

Let $S \in \Im^1_2(M_n)$ be tensor structure on the differentiable manifold M_n of class C^∞. Tensor field $t \in \Im^1_q(M_n)$ is called pure tensor field [1] with respect to the S-structure if it satisfies.

$$S^m_{j_1 j} t^{i_1}_{m j_2 \dots j_q} = \dots = S^m_{j_q j} t^{i_1}_{j_1 \dots j_{q-1} m} = S^{i_1}_{mj} t^m_{j_1 \dots j_q} = \overset{*}{t}{}^{i_1}_{j_1 i j_2 \dots j_q},$$
$$S^m_{i j_1} t^{i_1}_{m j_2 \dots j_q} = \dots = S^m_{i j_q} t^{i_1}_{j_1 \dots j_{q-1} m} = S^{i_1}_{im} t^m_{j_1 \dots j_q} = \overset{*}{t}{}^{i_1}_{i j_1 j_2 \dots j_q} \tag{1}$$

Now, let us consider the tensor bundle $T^1_q(M_n) = \bigcup_{Q \in M_n} T^1_q(Q)$ of type (1, q). Let $t^1_q(M_n) = \bigcup_{Q \in M_n} t^1_q(Q)$ be the subbundle of pure tensor field with respect to the S-structure, where $t^1_q(Q)$ is subspace of the pure tensors of type (1, q) at the point $Q \in M_n$ [2]. The components of the complete lift ${}^cS$ are given by

$${}^cS^j_{k_1 k_2} = S^j_{k_1 k_2}, \quad {}^cS^{\bar{j}}_{k_1 k_2} = \left(\partial_m S^{j_1}_{k_1 k_2}\right) t^m_{j_1 \dots j_q} - \sum_{a=1}^{q} \left(\partial_{j_a} S^m_{k_1 k_2}\right) t^{i_1}_{j_1 \dots m \dots j_q}, \tag{2}$$
$${}^cS^{\bar{j}}_{\bar{k} l} = S^{i_1}_{\ell_1 \ell} \delta^{k_1}_{j_1} \delta^{k_2}_{j_2} \dots \delta^{k_q}_{j_q}, \quad {}^cS^{\bar{j}}_{k \bar{\ell}} = S^{i_1}_{k m_1} \delta^{\ell_1}_{j_1} \delta^{\ell_2}_{j_2} \dots \delta^{\ell_q}_{j_q}$$

all the other being zero, with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}\}$, where $x^{\bar{k}} = t^{\ell_1}_{k_1 \dots k_q}$, $x^{\bar{\ell}} = t^{m_1}_{\ell_1 \dots \ell_q}$ [2]. Yano-Ako operator which is defined by S-structure was applied to the pure tensor field $t \in T^1_q(M_n)$ of type (1, q) and was obtained tensor field of type (1, q +2):

$$(\Phi^S t)(X, Y, X_1, X_2, ..., X_q) = (-L_{t(X_1, ..., X_q)} S)(X, Y) + t((L_{X_1} S)(X,Y), X_2, ..., X_q) + ... + t(X_1, X_2, ..., (L_{X_q} S)(X,Y)), \quad (3)$$

$\forall X_1, X_2, ..., X_q, X, Y \in T^1_0(M_n)$ or on the natural frame $\{\partial_i\}$ of coordinate neighborhood $U \subset M_n$ was obtained [1]:

$$\Phi^S_{kj} t^h_{i_1...i_q} = S^a_{kj}\partial_a t^h_{i_1...i_q} - t^a_{i_1...i_q}\partial_a S^h_{kj} - S^h_{kj}\partial_k t^a_{i_1...i_q} - S^h_{ka}\partial_j t^a_{i_1...i_q} + \sum_{\lambda=1}^{q} t^h_{i_1...a...i_q}\partial_{i_\lambda} S^a_{kj} \quad (4)$$

2. A FORMULA CONCERNING WITH THE YANO-AKO OPERATOR

Theorem 2.1. Let $\Phi^S_{ij} t^{i_1}_{j_1...j_q}$ be tha Yano-Ako operator. Then

$$\upsilon^i \omega^j \Phi^S_{ij} t^{i_1}_{j_1...j_q} = L_{S(V,W)} t^{i_1}_{j_1...j_q} - \upsilon^m S^{i_1}_{ml} L_W t^{\ell}_{j_1...j_q} - \omega^j S^{i_1}_{mj} L_V t^m_{j_1...j_q} \quad (5)$$

for $\forall V = \upsilon^i\partial_i$, $W = \omega^j\partial_j$, where L_V is the Lie derivative with respect to the vector field V.

Proof. If we consider equation (1), operation (4) is written as

$$\Phi^S_{ij} t^{i_1}_{j_1...j_q} = S^m_{ij}\partial_m t^{i_1}_{j_1...j_q} - \partial_i \overset{*}{t}{}^{i_1}_{j_1 j j_2...j_q} - \partial_j \overset{*}{t}{}^{i_1}_{i j_1 j_2...j_q} + \sum_{a=1}^{q}\left(\partial_{j_a} S^m_{ij}\right) t^{i_1}_{i_1...m...i_q} + \left(\partial_i S^{i_1}_{mj} + \partial_j S^{i_1}_{im} - \partial_m S^{i_1}_{ij}\right) t^m_{i_1...i_q} \quad (6)$$

Then, (6) can be rewritten as

$$\begin{aligned} \upsilon^i\left(\Phi^S_{ij} t^{i_1}_{j_1...j_q}\right) = {} & \upsilon^i S^m_{ij}\partial_m t^{i_1}_{j_1...j_q} - \partial_j\left(\upsilon^i \overset{*}{t}{}^{i_1}_{i j_1 j_2...j_q}\right) \\ & + \sum_{a=1}^{q}\partial_{j_a}\left(\upsilon^i S^m_{ij}\right) t^{i_1}_{j_1...m...j_q} + \left[\partial_j\left(\upsilon^i S^{i_1}_{im}\right) - \partial_m\left(\upsilon^i S^{i_1}_{ij}\right)\right] t^m_{j_1...j_q} \\ & + \left(\partial_j \upsilon^i\right) \overset{*}{t}{}^{i_1}_{i j_1...j_q} - \sum_{a=1}^{q}\left(\partial_{j_a}\upsilon^i\right) S^m_{ij} t^{i_1}_{j_1...m...j_q} - \left(\partial_j \upsilon^i\right) S^{i_1}_{im} t^{i_1}_{j_1...j_q} \\ & + \left(\partial_m \upsilon^i\right) S^{i_1}_{ij} t^m_{j_1...j_q} + \upsilon^i\left(\partial^i S^{i_1}_{mj}\right) t^m_{j_1...j_q} - \upsilon^i \partial_i \overset{*}{t}{}^{i_1}_{j_1 j j_2...j_q} \end{aligned} \quad (7)$$

$$= \Phi_j^{S(V)} t^{i_1}_{j_1 \cdots j_q} - \left(\upsilon^i \partial_i t^{*i_1}_{j_1 i j_2 \cdots j_q} + \sum_{a=1}^{q} \left(\partial_{ja} \upsilon^m \right) t^{*i_1}_{j_1 \hat{j} j_2 \ldots m \ldots j_q} \right)$$
$$+ \left(\partial_m \upsilon^i \right) S^{i_1}_{ij} t^m_{j_1 \cdots j_q} + \upsilon^i \left(\partial_i S^{i_1}_{mj} \right) t^m_{j_1 \cdots j_q} ,$$

where $\hat{j}$ shows dismiss of j in the above sum. To the Lie derivative, we write

$$L_V S^{i_1}_{mj} = \upsilon^i \partial_i S^{i_1}_{mj} + \left(\partial_m \upsilon^i \right) S^{i_1}_{ij} + \left(\partial_j \upsilon^i \right) S^{i_1}_{mi} - \left(\partial_m \upsilon^{i_1} \right) S^i_{mj}$$

or

$$\left(\partial_m \upsilon^i \right) S^{i_1}_{ij} + \upsilon^i \partial_i S^{i_1}_{mj} = L_V S^{i_1}_{mj} - \left(\partial_j \upsilon^i \right) S^{i_1}_{mi} + \left(\partial_m \upsilon^{i_1} \right) S^i_{mj} . \tag{8}$$

From equations (7) and (8), we find

$$\upsilon^i \Phi^S_{ij} t^{i_1}_{j_1 \cdots j_q} = \Phi_j^{S(V)} t^{i_1}_{j_1 \cdots j_q} - \left(\upsilon^i t^{*i_1}_{j_1 i j_2 \cdots j_q} + \sum_{a=1}^{q} \left(\partial_{ja} \upsilon^m \right) t^{*i_1}_{j_1 \hat{j} j_2 \cdots j_q} \right. \tag{9}$$
$$\left. + \left(\partial_j \upsilon^i \right) t^{*i_1}_{j_1 i j_2 \cdots j_q} - \left(\partial_i \upsilon^{i_1} \right) t^{*i}_{j_1 j j_2 \cdots j_q} \right) + \left(L_V S^{i_1}_{mj} \right) t^m_{i_1 \ldots i_q}$$
$$= \Phi_j^{S(V)} t^{i_1}_{j_1 \cdots j_q} - L_V t^{*i_1}_{j_1 j j_2 \cdots j_q} + \left(L_V S^{i_1}_{mj} \right) t^m_{i_1 \ldots i_q}$$
$$= \Phi_j^{S(V)} t^{i_1}_{j_1 \cdots j_q} - S^{i_1}_{mj} L_V t^m_{j_1 \cdots j_q}$$

where $\Phi^{S(V)}$ is Tachibana operator which is defined for affinor S(V) [3]. Similarly, the operation of contraction is written for the Tachibana operator, as

$$\omega^i \Phi_j^{S(V)} t^{i_1}_{j_1 \cdots j_q} = L_{S(V,W)} t^{i_1}_{j_1 \cdots j_q} - (S(V))^{i_1}_m L_V t^m_{j_1 \cdots j_q} . \tag{10}$$

From (9) and (10), we find

$$\upsilon^i \omega^j \Phi^S_{ij} t^{i_1}_{j_1 \cdots j_q} = L_{S(V,W)} t^{i_1}_{j_1 \cdots j_q} - S^{i_1}_{ml} \omega^m L_W t^{\ell}_{j_1 \cdots j_q} - \omega^j S^{i_1}_{mj} L_V t^m_{j_1 \cdots j_q} \tag{11}$$

3. LIFT ON THE CROSS-SECTION

Let us consider the tensor bundle of $T^1_q(M_n)$ with a natural projection $\pi : T^1_q(M_n) \to M_n$. If a differentiable mapping $\sigma : M_n \to T^1_q(M_n)$ satisfies $\pi o \sigma = id_{M_n}$, then σ is called a cross-section of $T^1_q(M_n)$, where id_{M_n} is the identity mapping on M_n. It is obvious that the cross-section on M_n defines a tensor field $t^{i_1}_{j_1 \cdots j_q}$ of type (1, q). Since the rank of the differential of the mapping σ is n and σ is injective, the cross-section of

$T^1_q(M_n)$ is submanifold of $T(M_n)$ with respect to induced topology, which is diffeomorphic to M_n. We will investigate the complete lift of a tensor S^i_{jk} along a pure submanifold defined by the cross-section.

The complete lift of a vector field $V = (\upsilon^i) \in T^1_0(M_n)$ to the tensor bundle $T^1_q(M_n)$ with respect to the coordinate neighborhood $\pi^{-1}(U) \subset T^1_q(M_n)$ was obtained in [4] as

$$ {}^cV = \left({}^cV^j \ , {}^cV^{\bar{j}} \right) = \left(\upsilon^i, L_V \alpha \right), \tag{12} $$

$\alpha \in T^q_1(U)$; $i = 1, ..., n$; $\bar{i} = n + 1, ... n + n^{1+q}$ where α can be considered as a differentiable function on the space $T^1_q(M_n)$ in the usual way by contraction $\alpha = \alpha(t)$. Particularly, if we get $\alpha = - t^{i_1}_{j_1 \cdots j_q}$, then the complete lift of V to $T^1_q(M_n)$ in the coordinate neighborhood $\pi^{-1}(U)$ with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}\}$, $x^{\bar{j}} = t^{i_1}_{j_1 \cdots j_q}$, is given by

$$ {}^cV = \left({}^cV^j \ , {}^cV^{\bar{j}} \right) = \left(\upsilon^i, \ t^m_{(j)} \partial_m \upsilon^{i_1} - \sum_{m=1}^{q} t^{i_1}_{j_1 \ldots m \ldots j_q} \partial_{j\mu} \upsilon^m \right). \tag{13} $$

Let us consider the cross-section of $T^1_q(M_n)$ defined by the tensor field $t^{i_1}_{j_1 \cdots j_q}(x^i)$. This cross-section equation is written as

$$ \bar{x}^J = \bar{x}^J(x^j), \quad J = 1, ..., n + n^{1+q} $$

or

$$ \left. \begin{aligned} \bar{x}^j &= x^j \\ \bar{x}^{\bar{j}} &= t^{i_1}_{j_1 \cdots j_q}(x^j). \end{aligned} \right\} $$

It is obvious that the system

$$ \left. \begin{aligned} B_i &= \left\{ \partial_i \bar{x}^A \right\} = \left\{ B^h_i B^{\bar{h}}_i \right\} = \left\{ \delta^h_i \partial_i t^{i_1}_{j_1 \cdots j_q} \right\} = \delta^h_i \partial_h + \partial_i t^{i_1}_{j_1 \cdots j_q} \partial_{\bar{h}} \\ C_{\bar{i}} &= \left\{ \partial_{\bar{i}} \bar{x}^A \right\} = \left(C^h_{\bar{i}} C^{\bar{h}}_{\bar{i}} \right) = \left(0, \ \delta^{l_1}_{j_1} \ldots \delta^{l_q}_{j_q} \delta^{i_1}_{h_1} \right) = \delta^{l_1}_{j_1} \ldots \delta^{l_q}_{j_q} \delta^{i_1}_{h_1} \partial_{\bar{h}} \end{aligned} \right\} $$

defines a frame along the cross-section. B_i and $C_{\bar{i}}$, $i = 1, ..., n$; $\bar{i} = n + 1$, ..., $n + n^{1+q}$, span the tangent plane of $T^1_q(M_n)$ and they are tangent to the cross-section and the fibre, respectively.

Using (13) and ${}^cV^A = \tilde{V}^i B^A_i + \tilde{V}^{\bar{j}} C^A_{\bar{i}}$, we have

$$\left.\begin{aligned} &\upsilon^i \partial_i x^{\bar h} + \left(t^m_{(j)} \partial_m \upsilon^{i_1} - \sum_{\mu=1}^{q} t^{i_1}_{j_1 \dots m \dots j_q} \partial_{j\mu} \upsilon^m \right) \partial_{\bar i} x^h = \tilde V^i B_i^{\bar h} + \tilde V^{\bar i} C_{\bar i}^{\bar h} \\ &\upsilon^i \partial_i x^h + \left(t^m_{(j)} \partial_m \upsilon^{i_1} - \sum_{\mu=1}^{q} t^{i_1}_{j_1 \dots m \dots j_q} \partial_{j\mu} \upsilon^m \right) \partial_{\bar i} x^h = \tilde V^i B_i^h + \tilde V^{\bar i} C_{\bar i}^h . \end{aligned}\right\}$$

Therefore, we obtain

$$\tilde V^i = \upsilon^i$$

$$\tilde V^{\bar h} = - L_V t^{i_1}_{j_1 \dots j_q} ,$$

that is the complete lift ${}^cV$ of V with respect to the frame (B, C) along the cross-section is written as

$${}^cV = \left({}^cV^j , {}^cV^{\bar j} \right) = \left(\upsilon^j , -L_V t^{i_1}_{j_1 \dots j_q} \right) . \tag{14}$$

We define the complete lift ${}^cS$ of a tensor $S \in \mathfrak{T}^1_q(M_n)$ along the pure cross-section of $T^1_q(M_n)$ by

$${}^c(S(V, W)) = {}^cS({}^cV, {}^cW). \tag{15}$$

The equality (15) can be written as

$${}^c(S(V, W))^I = {}^cS^I_{JK} {}^cV^J {}^cW^K \tag{16}$$

by using coordinates. If we take I = i in (16), we have

$$\begin{aligned} S^i_{jk} \upsilon^j \omega^k = (S(V, W))^i = {}^cS^I_{JK} {}^cV^J {}^cW^K &= {}^cS^i_{jk} {}^cV^j {}^cW^k + {}^cS^i_{\bar j k} {}^cV^{\bar j} {}^cW^k \\ &+ {}^cS^i_{j \bar k} {}^cV^j {}^cW^{\bar k} + {}^cS^i_{\bar j \bar k} {}^cV^{\bar j} {}^cW^{\bar k} \end{aligned}$$

Then, we obtain

$${}^cS^i_{jk} = S^i_{jk} , \ {}^cS^i_{\bar j k} = {}^cS^i_{j \bar k} = {}^cS^i_{\bar j \bar k} = 0 . \tag{17}$$

If we take $I = \bar i$ in the equality (16), we have

$$\begin{aligned} {}^c(S(V, W))^{\bar i} = {}^cS^{\bar i}_{jk} {}^cV^J {}^cW^K &= {}^cS^{\bar i}_{jk} {}^cV^j {}^cW^k + S^{\bar i}_{\bar j k} {}^cV^{\bar j} {}^cW^k \\ &+ {}^cS^{\bar i}_{j \bar k} {}^cV^j {}^cW^{\bar k} + {}^cS^{\bar i}_{\bar j \bar k} {}^cV^{\bar j} {}^cW^{\bar k} \end{aligned} \tag{18}$$

Now, let us find solutions which are ${}^cS^{\bar i}_{jk}$, ${}^cS^{\bar i}_{\bar j k}$, ${}^cS^{\bar i}_{j \bar k}$, ${}^cS^{\bar i}_{\bar j \bar k}$ of the equation (18). For this purpose, taking account of (13), we have

$$L_{S(VW)} t^{i_1}_{j_1 \ldots j_q} = \upsilon^i \omega^j \overset{S}{\Phi}_{ij} t^{i_1}_{j_1 \ldots j_q} + S^{i_1}_{ml} \upsilon^m L_W t^{\ell}_{j_1 \ldots j_q} + \omega^j S^{i_1}_{mj} L_V t^m_{j_1 \ldots j_q} \tag{19}$$

From (14) and (19), we get

$$-^c(S(V, W))^{\bar{i}} = \upsilon^i \omega^j \overset{S}{\Phi}_{ij} t^{i_1}_{j_1 \ldots j_q} + \upsilon^m S^{i_1}_{ml} \delta^{l_1}_{j_1} \ldots \delta^{l_q}_{j_q} L_W t^l_{l_1 \ldots l_q} +$$
$$+ \omega^m S^{i_1}_{lm} \delta^{l_1}_{j_1} \ldots \delta^{l_q}_{j_q} L_V t^l_{l_1 \ldots l_q} = {}^cV^i {}^cW^j \overset{S}{\Phi}_{ij} t^{i_1}_{j_1 \ldots j_q} -$$
$$- {}^cV^m S^{i_1}_{ml} \delta^{l_1}_{j_1} \ldots \delta^{l_q}_{j_q} {}^cW^{\bar{l}} - {}^cV^m S^{i_1}_{lml} \delta^{l_1}_{j_1} \ldots \delta^{l_q}_{j_q} {}^cW^{\bar{l}} \tag{20}$$

Then, from (18) and (20), we obtain

$$ {}^cS^{\bar{j}}_{ij} = - \overset{S}{\Phi}_{ij} t^{i_1}_{j_1 \ldots j_q} , \tag{21}$$

$$ {}^cS^{\bar{j}}_{m\bar{l}} = S^{i_1}_{ml} \delta^{l_1}_{j_1} \ldots \delta^{l_q}_{j_q} ,$$

$$ {}^cS^{\bar{j}}_{\bar{l}m} = S^{i_1}_{lm} \delta^{l_1}_{j_1} \ldots \delta^{l_q}_{j_q} , ,$$

$$ {}^cS^{\bar{j}}_{\bar{i}\bar{k}} = 0$$

Thus (17) and (21) are the complete lift of the tensor structure $S \in T^1_q(M_n)$ along the pure cross-section of $T^1_q(M_n)$. In particular, if the pure cross-section is integrable, that is $\partial_i t^{i_1}_{j_1 \ldots j_q} = 0$, hence we find formulea (2) from (17) and (21).

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