Commun. Fac. Sci. Univ. Ank. Series A1
V. 47. pp. 167-173 (1998)

CELLULAR FOLDING

E.M. ELKHOLY*, R.M. SHAHIN**

* *Department of Mathematics, Faculty of Sciences, Tanta University, Tanta, EGYPT.*
** *Department of Mathematics, Faculty of Education, Tanta University, Kafr El-Sheikh, EGYPT.*

(Received March 20, 1998; Accepted Dec. 18, 1998)

ABSTRACT

In this paper we introduce the notion of cellular and neat cellular foldings on a category of complexes equiped with cellular subdivision such that each closed n-cell is homeomorphic to a closed Euclidean n-cell. Then we obtained the necessary and sufficient conditions for a cellular map to be a cellular folding and a neat cellular folding respectively.

1. INTRODUCTION

Let K and L be directed complexes and f: $|K| \to |L|$ be continuous function. Then f: $K \to L$ is a cellular function if

(1) for each directed cell $\sigma \in K$, $f(\sigma) = \pm\tau$ where τ is a directed cell in L,

(2) $\dim(f(\sigma)) \leq \dim(\sigma)$, [4].

Let K and L be complexes of the same dimension n and K be equipped with finite cellular subdivision such that each closed n-cell is homeomorphic to a closed Euclidean n-cell.

A cellular map Φ: $K \to L$ is **a cellular folding** iff Φ satisfies the following:

(i) For each i-cell $e^i \in K$, $\Phi(e^i)$ is an i-cell in L. ie. Φ maps i-cells to i-cells.

(ii) If $\bar{e}$ contains n vertices, then $\overline{\Phi(e)}$ must contains n distinct vertices.

In the case of directed complexes it is also required that Φ maps directed i-cells of K to i-cells of L but of the same orientation.

A cellular folding Φ: K $\to$ L is a **neat cellular folding** if $L^n - L^{n-1}$ consists of a single n-cell. Int L.

The set of complexes together with the neat cellular foldings form a category which is a subcategory of the category of cellular foldings and we denote it by N(K,L). Thus if N(K,L) $\neq \Phi$, then dim L $\geq$ dim K.

Throughout this paper, we use the term complex to mean a complex equipped with celular subdivision such that each closed n-cell is homeomorphic to a closed Euclidean n-cell.

2. CHAIN MAPS AND CELLULAR FOLDING

The next theorem gives the necessary and sufficient condition for a cellular map to be a cellular folding.

THEOREM 1. Let K and L be complexes of the same dimension n and Φ: K $\to$ L be a cellular map such that $\Phi(K) = L \neq K$. Then Φ is a cellular folding iff the map $\Phi_p: C_p(K) \to C_p(L)$ between chain complexes $(C_p(K), \partial_p)$, $(C_p(L), \partial_p')$ is a chain map.

PROOF. Let Φ be a cellular folding, then it is a cellular map and we can define a homomorphism $\Phi_p: C_p(K) \to C_p(L)$ by

$$\Phi_p(\sigma) = \begin{cases} \Phi(\sigma) \text{ if } \Phi(\sigma) \text{ is a p-cell in L,} \\ \Phi \quad \text{ if } \dim(\Phi(\sigma)) < p \text{ , [4]} \end{cases}$$

and since a cellular folding maps p-cells to p-cells, $\Phi_p(\sigma_\lambda)$ is a p-cell in L for all λ.

Thus for a p-chain $C = a_1\sigma_1^p + a_2\sigma_2^p + \dots + a_m\sigma_m^p \in C_p(K)$ where $a_1's \in Z$ and $\sigma_1's$ are p-cells in K,

$$\Phi_p(C) = \Phi_p(a_1\sigma_1^p + a_2\sigma_2^p + \dots + a_m\sigma_m^p)$$

$$\Phi_p(C) = \Phi_p(a_1\sigma_1^p) + \Phi_p(a_2\sigma_2^p) + \dots + \Phi_p(a_m\sigma_m^p)$$

$$= a_1\Phi_p(\sigma_1^p) + a_2\Phi_p(\sigma_2^p) + \dots + a_m\Phi_p(\sigma_m^p) \in C_p(L).$$

Now since the closure of both σ_λ^p and $\Phi(\sigma_\lambda^p)$ has the same number of distinct vertices, then $\Phi_{p-1} \circ \partial_p = \partial_p' \circ \Phi_p$, where $\partial_p: C_p(K) \to C_{p-1}(K)$

and ∂'_p: $C_p(L) \to C_{p-1}(L)$ are the boundary operators, that is to say the following diagram commutes

$$\begin{array}{ccc} C_p(K) & \xrightarrow{\Phi_p} & C_p(L) \\ \downarrow{\scriptstyle \partial_p} & & \downarrow{\scriptstyle \partial'_p} \\ C_{p-1}(K) & \xrightarrow[\Phi_{p-1}]{} & C_{p-1}(L) \end{array}$$

and hence Φ is a chain map.

Conversly, suppose Φ is not a cellular folding, then there exists a j-cell σ in K such that $\Phi(s)$ is a m-cell in L, where $m \neq j$. Since Φ_p is a homomorphism from the p-th chain of K to the p-th chain of L, then

$$\Phi_j \left(\sum_{i=1}^{n-1} \lambda_i \sigma_i^{(j)} + \lambda_n \sigma\right) = \sum_{i=1}^{n-1} \lambda_i \Phi(\sigma_i^{(j)}) + \lambda_n \Phi(\sigma)$$

but $\Phi(\sigma)$ is not a j-cell, then Φ_j cannot be a j-chain map and hence our assumption is false and we have the result.

2.1. Examples

1- Let K be a complex such that |K| is the infinite stip $\{(x,y): -\infty < x < \infty, 0 \leq y \leq 2\}$ equipped with an infinite number of 2-cells such that the closure of each 2-cell consists of four 0-cells and four 1-cells, see Fig. 1.

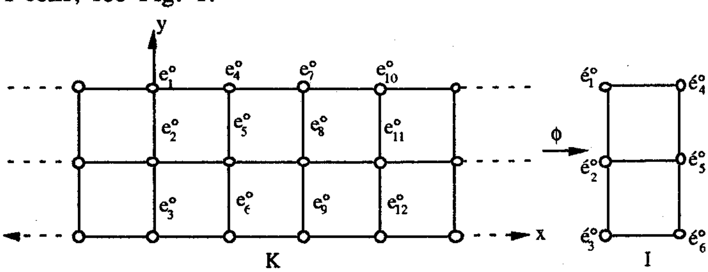

Fig. 1

Let L be a complex with six 0-cells, seven 1-cells and two 2-cells. The cellular map Φ: $K \to L$ defined by

$$\Phi(e_n^o) = e'^o_m \begin{cases} \text{where } m = 1,2,\ldots,6, & \text{and} \\ \quad n - m \text{ is a multiple of } 6 \end{cases}$$

$$\Phi(e_1^2) = \begin{cases} e'^2_1 \text{ if } i \text{ is odd} \\ e'^2_2 \text{ if } i \text{ is even} \end{cases}$$

This map is a cellular folding.

2- Consider a complex K such that $|K| = S^2$ with cellular subdivision consisting of two 0-cells, four 1-cells and four 2-cells. Let Φ: K $\rightarrow$ K be a cellular map defined by

$$\Phi(e_1^o, e_2^o) = (e_1^o, e_2^o)$$
$$\Phi(e_2^1, e_4^1) = (e_1^1, e_2^1)$$
$$\Phi(e_n^2, e_1^2) = e_1^2 \quad \text{for} \quad n = 1,2,3,4.$$

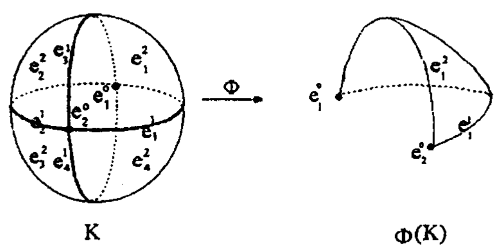

Fig. 2

This map is a cellular folding with image consisting of two 0-cells, two 1-cells and a single 2-cell, see Fig. 2.

3- Consider a complex K such that |K| is a tours with cellular subdivision consisting of three 0-cells, six 1-cells and three 2-cells.

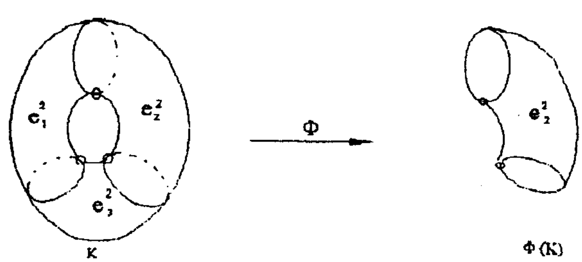

Fig. 3

Any cellular map Φ: K $\to$ K which has two vertices in the image is not a cellular folding since Φ_1 is not a chain map in this case.

4- Consider a complex K such that |K| is a tours with cellular subdivision consisting of four 0-cells, eight 1-cells and four 2-cells.

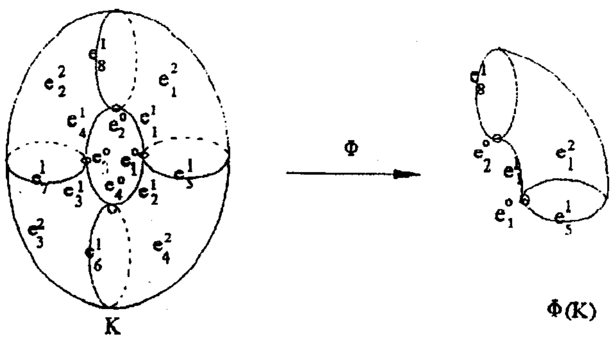

Fig. 4

A cellular map Φ: K $\to$ K defined by

$$\Phi(e_1^0, e_2^0, e_3^0, e_4^0) = (e_1^0, e_2^0, e_3^0, e_4^0)$$
$$\Phi(e_1^1, e_2^1, \ldots, e_8^1) = (e_1^1, e_1^1, e_1^1, e_1^1, e_5^1, e_8^1, e_5^1, e_8^1)$$
$$\Phi(e_n^2) = e_1^2 \quad \text{for} \quad n = 1,2,3,4.$$

This map is a cellular folding with image consisting of two 0-cells, three 1-cells and a single 2-cell.

5- Consider a cell-complex K such that $|K| = S^2$ with cellular subdivision consisting of four 0-cells, six 1-cells and four 2-cells, see Fig. 5.

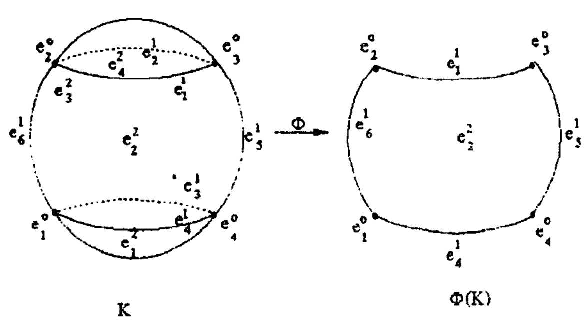

Fig. 5

Let Φ: K $\to$ K be a cellular map defined by

$\Phi(e_1^o, e_2^o, e_3^o, e_4^o) = (e_1^o, e_2^o, e_3^o, e_4^o)$

$\Phi(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1) = (e_1^1, e_1^1, e_4^1, e_4^1, e_5^1, e_6^1)$

$\Phi(e_n^2) = e_2^2$, n = 1,2,3,4.

This map is not cellular folding since $\overline{e_1^2}$, $\overline{\Phi(e_1^2)}$ does not contain the same number of vertices.

3. NEAT CELLULAR FOLDING

The following theorem gives necessary and sufficient condition for a cellular map to be a neat cellular folding.

THEOREM (2). If $\Phi \in N(K,L)$ such that $\Phi(K) = L \neq K$, then Φ is a neat cellular folding iff the map $\Phi_p : C_p \to C_p(L)$ between the chain complexes $(C_p(K),\partial_p)$, $(C_p(L),\partial_p')$ is a chain map and $H_p(K) \simeq \ker \Phi_p^*$, where $\Phi_p^*: H_p(K) \to H_p(L)$, $p \geq 1$ is the induced homomorphism.

PROOF. Assuming that Φ is a neat cellular folding, then it is a cellular folding and hence the map $\Phi_p : H_p(K) \to H_p(L)$ between the chain complexes $(C_p(K),\partial_p)$, $(C_p(L),\partial'_p)$ is a chain map. Now consider the induced homomorphism $\Phi_p^*: H_p(K) \to H_p(L)$, there is a short exact sequence:

$$0 \to \ker \Phi_p^* \xrightarrow{i^*} H_p(K) \xrightarrow{\Phi_p^*} \operatorname{Im} \Phi_p^*$$

where i^* is the induced homomorphism by the inclusion. Since Φ is surjective, we have $\operatorname{Im} \Phi_p^* \simeq H_p(L)$, but $H_p(L) = 0$ for neat cellular folding, hence the above sequence will take the form:

$$0 \to \ker \Phi_p^* \xrightarrow{i^*} H_p(K) \to 0 .$$

The exactness of this sequence implies that

$$H_p(K) \simeq \ker \Phi_p^*$$

Conversely, suppose Φ_p is a chain map between chain complexes and $H_p(K) \simeq \ker \Phi_p^*$ but Φ is not neat, then $L^n - L^{n-1}$ consists of more than one n-cells. Thus

$$H_o(L) \simeq Z^j \quad , \quad H_p(L) = 0 \ , \text{ for } p = 1,2,\ldots,n$$

and

$$H_p(K) \simeq H_p(L) \oplus \ker \Phi_p^* \not\simeq \ker \Phi_p^* \quad \text{for } p = 0$$

hence the assumption is false and Φ is neat.

It should be noted that examples (2) and (4) are neat cellular foldings.

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