Commun. Fac. Sci. Univ. Ank. Series A1
V. 47. pp. 111-120 (1998)

THE SEQUENCES OF THE SHEAVES OF HOMOTOPY GROUPS

AYHAN ŞERBETÇİ

Ankara University, Faculty of Sciences, Department of Mathematics, Tandoğan, Ankara, TURKEY

(Received April 30, 1998; Accepted June 11, 1998)

ABSTRACT

In this paper, we construct the sequences of the sheaves of homotopy groups of a pair (X,A) and of a triple (X,A,B), where X is a connected, locally path connected, and semilocally simply connected topological space and A, B are both open connected, locally path connected, and semilocally simply connected subspaces of X with $B \subset A \subset X$. Furthermore, we prove the existence of a homomorphism induced by a continuous map f: $(X,A) \to (Y,B)$ (respectively, f: $(X,A,B) \to (Y,C,D)$) between the sequences of the sheaves of homotopy groups of the pairs (X,A) and (Y,B) (respectively, of the triples (X,A,B) and (Y,C,D)).

1. INTRODUCTION

The theory of sheaves, developed and applied to various topological problems, has recently been applied to algebraic geometry and to the theory of functions of several complex variables. Furthermore, for homology and cohomology groups in algebraic topology and algebraic geometry, the "coefficient" groups must often be taken locally; that is, should be sheaves of groups. This is one of the basic reasons for considering sheaves. Now let X be a connected, locally path connected and semilocally simply connected topological space with base point x_0. For $n \geq 1$, the nth homotopy group $\pi_n(X,x_0)$ of X is defined to be the homotopy classes of maps α: $(I^n,\dot{I}^n) \to (X,x_0)$ such that $\alpha(I^n) \subset X$ and $\alpha(\dot{I}^n) = x_0$, where $I^n = \{(t_1,...,t_n) \in \mathbb{R}^n \mid 0 \leq t_i \leq 1,\ i = 1,...,n\}$ is n-cube and $\dot{I}^n$ its boundary. Equivalently, $\pi_n(X,x_0)$ may be regarded as the homotopy classes of base-point preserving maps α: $(S^n,P_0) \to (X,x_0)$ such that $\alpha(S^n) \subset X$ and $\alpha(P_0) = x_0$, where S^n is n-sphere and $P_0 = (1,0,...,0)$. $\pi_n(X,x_0)$ is abelian for $n > 1$. Let $S_n(X)$ be the disjoint union of the nth homotopy groups obtained for each $x \in X$, i.e., $S_n(X) = \bigvee_{x \in X} \pi_n(X,x)$. Steenrod [5] showed that $S_n(X)$ is a bundle of coefficients. On the other hand, it is known that $S_n(X)$ is an algebraic sheaf. Indeed, for each $\sigma \in S_n(X)$, define a map φ: $S_n(X) \to X$ as, $\varphi(\sigma) = \varphi([\alpha]_x) = x$, where $x \in X$ and $\sigma = [\alpha]_x \in \pi_n(X,x)$. Let $x_0 \in X$ be arbitrary fixed point. Then there exists a path connected open neighborhood $W = W(x_0)$ of x_0 such that for any two points x and y in W, every pair of paths in W joining x to y are homotopic in X with endpoints held fixed since X is locally path connected and semilocally simply connected. If γ is a path from x_0 to x_1 for any $x_1 \in W$, then γ induces an isomorphism

$$(\gamma^*)_n: \pi_n(X,x_0) \to \pi_n(X,x_1)$$

for all n, given by $(\gamma^*)_n([\alpha_0 x]_{x_0}) = [\alpha_1]_{x_1}$ which depends only on the homotopy class of γ. Therefore we can define a mapping s: $W(x_0) \to S_n(X)$ with $s(x_1) = (\gamma^*)_n([\alpha_0]_{x_0}) = [\alpha_1]_{x_1}$. Clearly, s is well-defined. In fact, if γ_1 and γ_2 are two paths in W from x_0 to x_1, then they are homotopic in X with endpoints held fixed. Hence $(\gamma_1{}^*)_n = (\gamma_2{}^*)_n$. Moreover, $\varphi \circ s = 1_W$, and $s(x_0) = [\alpha_0]_{x_0} \in \pi_n(X,x_0)$. For each $x \in X$, all such sets $s(W) = \{s(x) = [\alpha]_x \in S_n(X): x \in W \subset X\}$ form a basis for the topology on $S_n(X)$. Indeed, let W_1, W_2 be any two path connected subsets of X and $\sigma \in s_1(W_1) \cap s_2(W_2)$. Then s_1, s_2 agree at $\varphi(\sigma) = \varphi([\alpha]_x) = x$, $(x \in W_1 \cap W_2)$ and by the definition of the mappings s_1, s_2, $s_1(W_1 \cap W_2) = s_2(W_1 \cap W_2)$. Therefore σ has a basic neighborhood $s_1(W_1 \cap W_2) = s_2(W_1 \cap W_2)$ inside $s_1(W_1) \cap s_2(W_2)$. It follows that s is a continuous mapping with respect to this topology. Let $x \in X$ be any point and $V = V(x) \subset X$ be an open neighborhood of x. Since $\varphi^{-1}(V) = \bigvee_{i \in I} s_i(V)$, φ is continuous with respect to the topology on $S_n(X)$. Moreover, $\varphi \mid s_i(V)$: $s_i(V) \to V$ is a homeomorphism for every $i \in I$. Then φ is locally topological. Therefore $(S_n(X),\varphi)$ is a sheaf on X. It is called the sheaf of nth homotopy groups. For $n = 1$, $S_n(X)$ is called the sheaf of fundamental groups [1,3]. It is a sheaf of abelian groups for $n > 1$. By the definition, $(S_n(X),\varphi)$ is also a covering space on X. For every $x \in X$, $\varphi^{-1}(x) = \pi_n(X,x)$ is called the stalk of the sheaf and denoted by $(S_n(X))_x$. The continuous map s: $W \to S_n(X)$ is called a section of $S_n(X)$ over W.

The collection of all sections of $S_n(X)$ over a fixed path connected open subset W of X is denoted by $\Gamma(W,S_n(X))$. The set $\Gamma(W,S_n(X))$ is a group with the pointwise addition. Furthermore, if $A \subset X$ is open, $S_n(X)\,|_A = \varphi^{-1}(A) = \bigvee_{x\in A} \pi_n(X,x)$, then $(S_n(X)\,|_A, \varphi \mid (S_n(X))\,|_A))$ is a subsheaf of $S_n(X)$.

If A is an open connected, locally path connected, and semilocally simply connected subspace of X, then nth relative homotopy group $\pi_n(X,A,x_0)$ of pair (X,A) with base point in A is defined as the set of all homotopy classes of the maps α: $(I^n, I^{n-1}, J^{n-1}) \to (X,A,x_0)$ such that $\alpha(I^n) \subset X$, $\alpha(I^{n-1}) \subset A$ and $\alpha(J^{n-1}) = x_0$, where $I^{n-1} = \{(t_1,\ldots,t_n) \in I^n \mid t_n = 0\}$, $J^{n-1} = \dot{I}^n - \mathrm{int}I^{n-1}$. Equivalently, we may represent $[\alpha] \in \pi_n(X,A,x_0)$ by a map α: $(E^n, S^{n-1},P_0) \to (X,A,x_0)$, where $E^n = \{(t_1,\ldots,t_n) \in \mathbb{R}^n \mid t_1^2 + \ldots + t_n^2 \leq 1\}$ is the unit n-cell, S^{n-1} is the (n–1)-sphere. $\pi_n(X,A,x_0)$ is an abelian group for $n > 2$. Let $S_n(X,A) = \bigvee_{x\in A} \pi_n(X,A,x)$. $S_n(X,A)$ is a set over A and the map φ: $S_n(X,A) \to A$ defined by $\varphi(\sigma) = \varphi([\alpha]_x) = x$ for any $x \in A$, $\sigma_x \in S_n(X,A)$, is onto. Similarly it is readily checked that $S_n(X,A)$ is an algebraic sheaf. For $n > 2$, $S_n(X,A)$ is the sheaf of abelian groups.

As indicated, $\pi_n(X,x_0)$ is only defined for $n \geq 1$, while $\pi_n(X,A,x_0)$ is only defined for $n \geq 2$. It is often useful to introduce $\pi_0(X,x_0)$ and $\pi_1(X,A,x_0)$. Let S^0 be the unit 0-sphere and P_0 the point (1,0). Then $\pi_0(X,x_0)$ is defined to be the set of homotopy classes of maps α: $(S^0,P_0) \to (X,x_0)$. We will not attempt to give this set any algebraic structure. It is clear that $\pi_0(X,x_0)$ has one element for each path component of X. Since X is path connected $\pi_0(X,x_0)$ is the pair (X,x_0). In a similar manner we use the symbol $\pi_1(X,A,x_0)$ to denote the set of all homotopy classes of maps α: $(E^1,S^0,P_0) \to (X,A,x_0)$. The "identity elements" of this set is the class of the constant map $E^1 \to x_0$ and is denoted by 0.

2. THE SEQUENCE OF THE SHEAVES OF HOMOTOPY GROUPS OF A PAIR (X,A)

Let X be a connected, locally path connected, and semilocally simply connected topological space and A be an open connected, locally path connected, and semilocally simply connected subspace of X. We can define a boundary operator on the sheaf of relative homotopy groups. In fact; for every $x_0 \in A$, if $[\alpha]_{x_0} \in S_n(X,A)$ with α: $(E^n, S^{n-1}, P_0) \to (X,A,x_0)$, then the homotopy class of $\alpha \mid S^{n-1} : (S^{n-1}, P_0) \to (A,x_0)$ depends only on that α, so that we may define ∂^*: $S_n(X,A) \to S_{n-1}(A)$ by $\partial^*([\alpha]_{x_0}) = [\alpha \mid S^{n-1}]_{x_0}$. Equivalently we may represent $[\alpha]_x \in S_n(X,A)$ by α: $(I^n, I^{n-1}, J^{n-1}) \to (X,A,x_0)$ and then we have $\alpha \mid I^{n-1}$: $(I^{n-1}, \dot{I}^{n-1}) \to (A,x_0)$ and $\partial^*([\alpha]_{x_0}) = [\alpha \mid I^{n-1}]_{x_0}$.

Lemma 1. The boundary operator ∂^*: $S_n(X,A) \to S_{n-1}(A)$ is a sheaf morphism.

Proof. Firstly, ∂^* is continuous. Let $x \in A \subset X$ and W(x) be a path connected open neighborhood of x in A. Let s: $W(x) \to S_n(X,A)$ be a section and $[\alpha]_x \in s(W)$. Then, s(W) is an open neighborhood of $[\alpha]_x$. If $\partial^*([\alpha]_x) = [\beta]_x = [\alpha \mid I^{n-1}]_x$, then there exists a section t: $W(x) \to S_{n-1}(A)$ and t(W) is an open neighborhood of $[\beta]_x$, and so for every $x \in W$, $\partial^*(s(x_1)) = \partial^*([\alpha_1]_{x_1}) = [\alpha_1 \mid I^{n-1}]_{x_1} \in t(W)$. Therefore $\partial^*(s(W)) \subset t(W)$ and ∂^* is continuous.

Secondly, ∂^* is a homomorphism: Let $[\alpha_1]_x$, $[\alpha_2]_x \in S_n(X,A)$ then it is clear that

$$\partial^*([\alpha_1]_x + [\alpha_2]_x) = \partial^*([\alpha_1 + \alpha_2]_x) = [(\alpha_1 + \alpha_2) \mid I^{n-1}]_x$$
$$= [\alpha_1 \mid I^{n-1}]_x + [\alpha_2 \mid I^{n-1}]_x = \partial^*([\alpha_1]_x) + \partial^*([\alpha_2]_x).$$

Finally, ∂^* is a stalk preserving map: φ_1 and φ_2 are the natural projections of $S_n(X,A)$ and $S_{n-1}(A)$, respectively. For every $[\alpha]_x \in S_n(X,A)$, $\varphi_1([\alpha]_x) = x$ and $(\varphi_2 \circ \partial^*)([\alpha]_x) = \varphi_2(\partial^*([\alpha]_x)) = \varphi_2([\alpha \mid I^{n-1}]_x) = \varphi_2([\beta]_x) = x$, hence $\varphi_1 = \varphi_2 \circ \partial^*$.

Lemma 2. Let the sheaves $S_n(X)$ and $S_n(Y)$ be given. A continuous map f: $X \to Y$ induces a stalk preserving sheaf homomorphism of sheaves f^*: $S_n(X) \to S_n(Y)$.

Proof. [1,3].

Lemma 3. Let the sheaves $S_n(X,A)$ and $S_n(Y,B)$ be given. A continuous map $f:(X,A) \to (Y,B)$ induces a stalk preserving sheaf homomorphism of sheaves $f^*: S_n(X,A) \to S_n(Y,B)$.

Proof. Similar to the proof of Lemma 2.

Lemma 4. For a map $\alpha: (E^n,S^{n-1},P_0) \to (X,A,x_0)$, $[\alpha] = 0$ in $\pi_n(X,A,x_0) \in S_n(X,A) \Leftrightarrow \alpha$ is homotopic, rel S^{n-1}, to a map into A.

Proof. [2, p. 448 Theorem 5.8].

Lemma 5. For a map $\alpha: (S^n,P_0) \to (X,x_0)$, $[\alpha] = 0$ in $\pi_n(X,x_0) \in S_n(X) \Leftrightarrow \alpha$ has an extension $\alpha': E^{n+1} \to X$.

Proof. [4, p. 36 Lemma 2.21].

We can now set up the sequence of the sheaves of homotopy groups of the pair (X,A). Let i: $(A,x_0) \to (X,x_0)$, j: $(X,x_0) \to (X,A)$, $x_0 \in A$, be the inclusion maps, then we have sheaf homomorphisms

$$i^*: S_n(A) \to S_n(X),\ j^*: S_n(X)\,|_A \to S_n(X,A)$$

and $\partial^*: S_n(X,A) \to S_{n-1}(A)$ by Lemmas 1, 2, and 3. We have a sequence

$$\begin{aligned} \ldots &\to S_{n+1}(X,A) \xrightarrow{\partial^*} S_n(A) \xrightarrow{i^*} S_n(X)\,|_A \xrightarrow{j^*} S_n(X,A) \to \ldots \\ &\to S_1(X)\,|_A \xrightarrow{j^*} S_1(X,A) \xrightarrow{\partial^*} S_0(A) \xrightarrow{i^*} S_0(X) \end{aligned}$$

called the sheaf sequence of homotopy groups of the pair (X,A).

Theorem 1. The sequence of the sheaves of homotopy groups of the pair (X,A) is exact.

(Note that in this exact sequence the last three maps are not sheaf homomorphism, but only set maps. The kernel of a set map between pointed sets is by definition the inverse image of the base point, i.e., they are just base-point preserving functions. Exactness in this context is given by the same condition: "the image of each map is the kernel of the next").

Proof. The proof of Theorem 1 is divided into six parts.

(1) $i^* \circ \partial^* = 0$. For if $[\alpha] \in S_{n+1}(X,A)$ with α: $(E^{n+1},S^n,P_0) \to (X,A,x_0)$, then $\alpha \mid S^n$: $(S^n,P_0) \to (A,x_0)$ represents $\partial^*([\alpha])$, and $i^*\partial^*([\alpha])$ is represented by $i \circ (\alpha \mid S^n)$ which has an extension to E^{n+1}. Thus, by Lemma 5 $i^*\partial^*([\alpha]) = 0$.

(2) $j^* \circ i^* = 0$. If $[\alpha] \in S_n(A)$ with α: $(I^n,\dot{I}^n) \to (A,x_0)$, then $j^*i^*([\alpha])$ is represented by $(j \circ i) \circ \alpha$: $(I^n,I^{n-1},J^{n-1}) \to (X,A,x_0)$, where $\alpha(I^n) \subset A$. Since I^n is contractible over itself to $(0,0,...,0)$, then it follows that. $j^*i^*([\alpha]) = 0$.

(3) $\partial^* \circ j^* = 0$. If $[\alpha] \in S_n(X) \mid_A$ with α: $(I^n,\dot{I}^n) \to (X,x_0)$, then $\partial^*j^*([\alpha])$ is represented by $j \circ (\alpha \mid I^{n-1})$: $(I^{n-1},\dot{I}^{n-1}) \to (A,x_0)$, but $\alpha(I^{n-1}) = x_0$, so that $\partial^*j^*([\alpha]) = 0$.

(1′) $\ker(i^*) \subset \mathrm{Im}(\partial^*)$. If $[\alpha] \in \ker(i^*)$, then α: $(S^n,P_0) \to (A,x_0)$ and $i \circ \alpha$ is nullhomotopic in X. By Lemma 5, α has an extension to α': $(E^{n+1},S^n) \to (X,A)$ which represents $[\alpha'] \in \pi_{n+1}(X,A,x_0)$ and $\partial^*([\alpha']) = [\alpha]$, i.e., $[\alpha] \in \mathrm{Im}(\partial^*)$.

(2′) $\ker(j^*) \subset \mathrm{Im}(i^*)$. If $[\alpha] \in \ker(j^*)$, then α: $(E^n,S^{n-1}) \to (X,x_0)$ and $j \circ \alpha$: $(E^n,S^{n-1},P_0) \to (X,A,x_0)$ is nullhomotopic. By Lemma 4, α is homotopic, rel S^{n-1}, to a map β: $E^n \to A$. The map β represents an element $[\beta] \in S_n(A)$ such that $i^*([\beta]) = [\alpha]$. Thus $[\alpha] \in \mathrm{Im}(i^*)$.

(3′) $\ker(\partial^*) \subset \mathrm{Im}(j^*)$. If $[\alpha] \in \ker(\partial^*)$, then α: $(E^n,S^{n-1},P_0) \to (X,A,x_0)$ and $\alpha \mid S^{n-1}$: $(S^{n-1},P_0) \to (A,x_0)$ is nullhomotopic. By the homotopy extension property, α is homotopic to a map α': $(E^n,S^{n-1}) \to (X,x_0)$. Then α' represents $[\alpha'] \in \pi_n(X,x_0)$ such that $j^*([\alpha']) = [\alpha]$, i.e., $[\alpha] \in \mathrm{Im}(j^*)$.

Corollary 1. For any connected, locally path connected, and semilocally simply connected topological space X, $S_n(X,X) = 0$ for all $n \geq 1$.

Now, if f: $(X,A) \to (Y,B)$ is a continuous map such that $f(X) \subset Y$, $f(A) \subset B$, then f induces homomorphisms

$$f_n^*: S_n(X)|_A \to S_n(Y)|_B,\ f_{n_1}^*: S_n(A) \to S_n(B)\ ,\ f_{n_2}^*: S_n(X,A) \to S_n(Y,B),$$

by Lemmas 1, 2 and 3.

We obtain a homomorphism between the sequence of the sheaves of homotopy groups of the pairs (X,A) and (Y,B):

$$\begin{array}{ccccccccc}
\cdots \to S_{n+1}(X,A) & \xrightarrow{\partial^*_{n_1+1}} & S_n(A) & \xrightarrow{i^*_{n_1}} & S_n(X)\,|_A & \xrightarrow{j^*_{n_1}} & S_n(X,A) \to \cdots \\
f^*_{n_2+1}\downarrow & & f^*_{n_1}\downarrow & & f^*_n\downarrow & & f^*_{n_2}\downarrow \\
\cdots \to S_{n+1}(Y,B) & \xrightarrow{\partial^*_{n_2+1}} & S_n(B) & \xrightarrow{i^*_{n_2}} & S_n(Y)\,|_B & \xrightarrow{j^*_{n_2}} & S_n(Y,B) \to \cdots
\end{array}$$

It is then a matter of straight forward verification from the definitions that the commutativity relations,

$$f^*_{n_1}\partial^*_{n_1+1} = \partial^*_{n_2+1}f^*_{n_2+1}\ ,\ f^*_n i^*_{n_1} = i^*_{n_2}f^*_{n_1+1}\ ,\ f^*_{n_2}j^*_{n_1} = j^*_{n_2}f^*_n$$

hold between the homomorphisms of the sequences of the sheaves of homotopy groups of the pairs (X,A) and (Y,B). We say that $(f^*_n, f^*_{n_1}, f^*_{n_2})$ is a homomorphism of the sequence of the sheaves of homotopy groups of the pair (X,A) into the sequence of the sheaves of homotopy groups of the pair (Y,B). If f^*_n, $f^*_{n_1}$, $f^*_{n_2}$ are isomorphisms for all n, we describe $(f^*_n, f^*_{n_1}, f^*_{n_2})$ as an isomorphism.

Therefore, we have the following theorem.

Theorem 2. Let f: (X,A) $\to$ (Y,B) be a continuous map. Then, there exists a homomorphism $(f^*_n, f^*_{n_1}, f^*_{n_2})$ between the sheaf sequences of homotopy groups of the pair (X,A) and (Y,B). Furthermore, if f is a topological map, then $(f^*_n, f^*_{n_1}, f^*_{n_2})$ is an isomorphism.

3. THE SEQUENCE OF THE SHEAVES OF HOMOTOPY GROUPS OF A TRIPLE (X,A,B)

Let X be a connected, locally path connected, and semilocally simply connected topological space and A, B be open connected, locally path connected, and semilocally simply connected subspaces of X such that $B \subset A \subset X$. Let k: (A,B) $\hookrightarrow$ (X,B), l: (X,B) $\hookrightarrow$ (X,A) be the inclusion maps, then we have sheaf homomorphisms

$$k^*: S_n(A,B) \to S_n(X,B),\ l^*: S_n(X,B) \to S_n(X,A).$$

by lemma 3. Furthermore, we have a boundary operator

$$\partial^*: S_{n+1}(X,A)\,|_B \to S_n(A,B)$$

which, is defined as the composite of

$$S_{n+1}(X,A) \to S_n(A) \text{ and } S_n(A)\,|_B \to S_n(A,B),$$

where the first map is the boundary operator of the sheaf sequence of homotopy groups of the pair (X,A). Then we have the sequence of the sheaves of homotopy groups of the triple (X,A,B)

$$\ldots \to S_{n+1}(X,A)\,|_B \xrightarrow{\partial^*} S_n(A,B) \xrightarrow{k^*} S_n(X,B) \xrightarrow{l^*} S_n(X,A)\,|_B \to \ldots$$

This sequence is called the sequence of the sheaves of homotopy groups of the triple (X,A,B).

Theorem 3. The sequence of the sheaves of homotopy groups of the triple (X,A,B) is exact.

Proof. This follows from Theorem 1, together with certain elementary facts. Firstly, that $l^* \circ k^* = 0$ follows from commutativity of the diagram

$$\begin{array}{ccc} S_n(A,B) & \to & S_n(X,B) \\ \downarrow & & \downarrow \\ S_n(A,A) & \to & S_n(X,A) \end{array}$$

of injections and corollary 1. Secondly, we will describe the proof that $\ker(k^*) = \mathrm{Im}(\partial^*)$. Consider the diagram

$$\begin{array}{ccccc} & & S_n(B) & & S_n(B) \\ & \partial^*_{n+1} & \downarrow i^*_{n_1} & i^*_n & \downarrow i^*_{n_2} \\ S_{n+1}(X,A)\,|_B & \to & S_n(A)\,|_B & \to & S_n(X)\,|_B \\ & \partial^* \searrow & \downarrow j^*_{n_1} & k^* & \downarrow j^*_{n_2} \\ & & S_n(A,B) & \to & S_n(X,B) \\ & & \downarrow \partial^*_{n_1} & & \downarrow \partial^*_{n_2} \\ & & S_{n-1}(B) & & S_{n-1}(B) \end{array}$$

Here ∂^*_{n+1}, i^*_n are homomorphisms of the sheaf sequence of the pair (X,A); $i^*_{n_1}$, $j^*_{n_1}$, $\partial^*_{n_1}$ relate to the pair (A,B); and $i^*_{n_2}$, $j^*_{n_2}$, $\partial^*_{n_2}$ relate to the pair (X,B). By definition, we have $\partial^* = j^*_{n_1}\partial^*_{n+1}$, and the following 'commutativity' relations obviously hold:

$$i^*_n i^*_{n_1} = i^*_{n_2}\ ,\ k^* j^*_{n_1} = j^*_{n_2} i^*_n\ ,\ \partial^*_{n_2} k^* = \partial^*_{n_1}\ .$$

Now let $[\alpha] \in S_{n+1}(X,A)|_B$. Then

$$k^*\partial^*([\alpha]) = k^* j^*_{n_1}\partial^*_{n+1}([\alpha]) = j^*_{n_2} i^*_n \partial^*_{n+1}([\alpha]) = 0,$$

since $i^*_n\partial^*_{n+1} = 0$. Thus $\mathrm{Im}(\partial^*) \subset \ker(k^*)$. Now let $[\alpha] \in S_n(A,B)$, and let $k^*([\alpha]) = 0$. Then $\partial^*_{n_1}([\alpha]) = \partial^*_{n_2} k^*([\alpha]) = 0$, so that by exactness, $[\alpha] = j^*_{n_1}([\beta])$, for any $[\beta] \in S_n(A)\ |_B$. Then $0 = k^*([\alpha]) = k^* j^*_{n_1}([\beta]) = j^*_{n_2} i^*_n([\beta])$, so that by exactness, $i^*_n([\beta]) = i^*_{n_2}([\gamma])$, for any $[\gamma] \in S_n(B)$. Thus

$$i^*_n([\beta] - i^*_{n_1}([\gamma])) = i^*_n([\beta] - i^*_{n_2}([\gamma])) = 0,$$

whence, by exactness, $[\beta] - i^*_{n_1}([\gamma]) = \partial^*_{n+1}([\delta])$, for any $[\delta] \in S_n(X,A)|_B$, and $[\alpha] = j^*_{n_1}([\beta]) = j^*_{n_1}([\beta] - i^*_{n_1}([\gamma])) = j^*_{n_1}\ \partial^*_{n+1}([\delta]) = \partial^*([\delta])$, since $j^*_{n_1}\ i^*_{n_1} = 0$. This shows that $\ker(k^*) \subset \mathrm{Im}(\partial^*)$ and so completes the proof of the assertion that $\ker(k^*) = \mathrm{Im}(\partial^*)$. Similarly routine diagram chasing completes the proof of remaining assertions.

Now let (X,A,B), (Y,C,D) be triples and f: (X,A,B) $\to$ (Y,C,D) be a continuous map such that $f(x) \subset Y$, $f(A) \subset C$, and $f(B) \subset D$. Clearly the map f induces maps f_1: (X,A) $\to$ (Y,C), f_2: (A,B) $\to$ (C,D), and f_3: (X,B) $\to$ (Y,D). Then f_1, f_2, f_3 induce homomorphisms

$f^*_{n_1}$: $S_n(X,A)|_B \to S_n(Y,C)|_D$, $f^*_{n_2}$: $S_n(A,B) \to S_n(C,D)$, $f^*_{n_3}$: $S_n(X,B) \to S_n(Y,D)$

Then we obtain a homomorphism $(f^*_{n_1}, f^*_{n_2}, f^*_{n_3})$ between the sheaf sequences of homotopy groups of the triples (X,A,B) and (Y,C,D)

$$\begin{array}{ccccccccc}
\ldots \to S_{n+1}(X,A)\,|_B & \to & S_n(A,B) & \to & S_n(X,B) & \to & S_n(X,A)\,|_B & \to \ldots \\
f^*_{n_1+1}\downarrow & & f^*_{n_2}\downarrow & & f^*_{n_3}\downarrow & & f^*_{n_1}\downarrow & \\
& & & & & i^*_{n_2} & & \\
\ldots \to S_{n+1}(Y,C)\,|_D & \to & S_n(C,D) & \to & S_n(Y,D) & \to & S_n(Y,C)\,|_D & \to \ldots
\end{array}$$

Therefore we give the following theorem. The proof of this theorem is similar to Theorem 2.

Theorem 4. Let f: (X,A,B) $\to$ (Y,C,D) be a continuous map. Then there exists a homomorphism $(f^*_{n_1}, f^*_{n_2}, f^*_{n_3})$ between the sequences of the sheaves of homotopy groups of the triples (X,A,B) and (Y,C,D). Furthermore, if f is a topological map, then $(f^*_{n_1}, f^*_{n_2}, f^*_{n_3})$ is an isomorphism.

REFERENCES

[1] BALCI, S., The Sheaf of The Fundamental Groups. Comm. Fac. Sci. Univ. Ankara Ser. A_1: Mathematiques, pp 59-65 (1982).

[2] BREDON, G.E., Topology And Geometry, Springer Verlag, New York (1993).

[3] CANPOLAT, N., Yüksek Homotopi Gruplarının Demeti Ve İlgili Karakterizasyonlar, PhD. Thesis, Ankara University (1982).

[4] HILTON, P.J., An Introduction to Homotopy Theory, Cambridge University Press (1961).

[5] STEENROD, N., The Topology of Fibre Bundles, Princeton University Press, New Jersey (1951).

[6] TENNISON, B.R., Sheaf Theory. Cambridge University Press (1975).

[7] WHITEHEAD, G.W., Elements of Homotopy Theory. Springer Verlag New York (1978).