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Research Paper

The Characterizations Of Null Quaternionic Curves In Minkowski 3-Space

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Abstract: In this study, we investigate some characterizations of null quaternionic curves in the Minkowski 3-Space. Also, we research the characterizations of Null quaternion curves in Minkowski 3-Space.

Key words: Null Quaternionic curves, Serret-Frenet formula, Helices, Slant Helices.

Minkowski 3-Uzayda Null Kuaterniyonik Eğrilerin Karekterizasyonları

Öz: Bu çalışmada, Minkowski 3-uzayda null kuaterniyonik eğrilerin bazı karakterizasyonları incelendi. Ayrıca, Minkowski 3- uzayda null kuaterniyonik eğrilerin karakterizasyonları çalışıldı.

Anahtar Kelimeler: Null Kuterniyonik Eğriler, Serret-Frenet Formülleri, Helis, Slant Helis

1. Introduction

The curve theory has been one of the most tried topics owing to existing a lot of implementation fields from geometry to the different twigs of science. Most mathematicians study the private curves like Bertrand and Mannheim curves. Lately, they have defined a new special curve named Smarandache curve in Minkowski 3-space time by Turgut and Yılmaz [1].

Quaternions were first defined by Irish Mathematician William Rowan Hamilton in 1843. Hamilton said that his adaptation was a generalization in which the real(scalar) axes were unaltered but completed by adding the vector in 1843. The Serret-Frenet formulas of the Null Quaternionic curves by Çöken and Tuna were defined for the Semi-Euclidean spaces, in 2015 [2-3]. The quaternionic curves in $\mathbb{E}^3$ and $\mathbb{E}^4$ were described by Baharathi and Nagaraj and they also carried out studies on differential geometry of space curves and with the help of their studies on quaternions, they introduced Frenet frames and formulae [4]. There are some curves which are particular in differential geometry. Having a crucial role, these curves satisfy some connections between their curvatures and torsions. Described by characteristic that tangent of curve makes a constant angle with a stable straight line called the axis of the general helix. Moreover, lately slant helix have been identify as a special curve [5]. Some characterizations of slant helices in Euclidean 3-space were analyzed by Kula and et al. [6].

In this study the characterizations of null quaternionic curves in Minkowski 3-space were analyzed. Also the characterizations of a null quaternionic curve to be helix, slant helix were given.

2. Preliminaries

In this part, we impart fundamental notions connected to the semi-real quaternions [2]. A series of semi-real quaternion is representable by

$$Q = \{q \mid q = ae_1 + be_2 + ce_3 + d; a, b, c, d \in \mathbb{R}\},$$

where

$$e_1, e_2, e_3 \in \mathbb{E}_1^3, h(e_i, e_i) = \varepsilon(e_i), 1 \leq i \leq 3$$

and

$$e_i \times e_i = -\varepsilon(e_i),$$
$$e_i \times e_j = \varepsilon(e_i)\varepsilon(e_i)e_k \in \mathbb{E}_1^3.$$

The cross product of semi-real quaternions for vectors p and q are described by

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$$p \times q = S_p S_q + S_p V_q + S_q V_p + h(V_p, V_q) + V_p \wedge V_q.$$

In this place, we used scalar and vector products defined in $\mathbb{E}_1^3$. To a semi real quaternion $q = ae_1 + be_2 + ce_3 + d$, conjugate αq of q and scalar product h_1 are denoted as $\alpha q = -ae_1 - be_2 - ce_3 + d$ and

$$h(p,q) = \frac{1}{2}[\varepsilon(p)\varepsilon(\alpha q)(p \times \alpha q) + \varepsilon(q)\varepsilon(\alpha p)(q \times \alpha p)]$$

respectively.

$\mathbb{E}_1^3$ is described as the space of null spatial quaternions $\left\{\gamma \in Q_{\mathbb{E}_1^3} \mid \gamma + \alpha\gamma = 0\right\}$ in an obvious manner,

$$\gamma(s) = \sum_{i=1}^{3} \gamma_i(s)\vec{e}_i \ , 1 \le i \le 3.$$

Let $\{l, n, u\}$ be the frenet trihedron for the differentiable null spatial quaternionic curve in $\mathbb{E}_1^3$, we took the e_2 as time-like vector. After that, Frenet formulae is

$$\begin{bmatrix} l' \\ n' \\ u' \end{bmatrix} = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & \tau \\ -\tau & -k & 0 \end{bmatrix} \begin{bmatrix} l \\ n \\ u \end{bmatrix},$$

where k and τ are first and second curvatures of null spatial quaternionic curve, respectively. Furthermore, $h(l,l) = h(n,n) = h(l,u) = h(n,u) = 0, h(l,n) = h(u,u) = 1$, n, l are null vectors; u is a space-like vector. In this place, the quaternion product is dedicated as

$$l \times n = -1 - u, \ \ n \times l = -1 + u, \ \ n \times u = -n, \ \ u \times n = n$$
$$u \times l = -l, \ \ l \times u = l, \ \ u \times u = -1, \ \ l \times l = n \times n = 0 \ .$$

3. Characterizations Of Null Quaternionic Curves In Minkowski 3-Space

In this part, we first examined some characterizations of null quaternionic curves in Minkowski 3-Space.

Definition 3.1.

The curve formed by making a fixed angle in a fixed direction is called a helix. İf ratio $\frac{k_1}{k_2}$ is constant, it means that the curves are helix [7].

Theorem 3.2.

Let's take an α null quaternionic curve in $\mathbb{E}_1^3$, at the time α is a common helix only if $\frac{k_1}{k_2}$ is fixed.

Proof. When α is treated as a common helix, the slope axis of the curve α is denoted as $sp\{A\}$. Note this

$$< l, A >= cons.$$

If we differentiate both sides of the equality above, by then we have

$$< l', A >= 0.$$

By using equations $l' = k.u$ and $A = \cos\theta.n + \sin\theta.u$,

$$< k.u, A >= 0$$
$$k < u, A >= 0, k \neq 0$$
$$< u, A >= 0.$$

If we differentiate both sides of the equality above;

$$< u', A > + < u, A' >= 0$$

Since derivative of the constant is zero, then;

$$< u', A >= 0$$
$$< -\tau l - kn, \cos\theta n + \sin\theta u > = 0$$
$$< -\tau l, \cos\theta n > + < -\tau l, \sin\theta u > + < -kn, \cos\theta n > + < -kn, \sin\theta u > = 0$$
$$-\tau\cos\theta < l, n > -\tau\sin\theta < l, u > -k\cos\theta < n, n > -k\sin\theta < n, u > = 0.$$

Here, since

$$< l, n >= < l, u >= < n, u >= 0 \text{ and } < n, n >= 1,$$

the following expression is obtained;

$$-k\cos\theta = 0$$
$$k\cos\theta = 0$$

the result is as follow:

Case i) $k = 0 \Rightarrow \cos\theta \neq 0$
Case ii) $k \neq 0 \Rightarrow \cos\theta = 0.$

Definition 3.3.

A unit speed curve α is dubed a slant helix if the function $< n(s), U >$ is constant, for a non-zero fixed vector field $U \in \mathbb{E}_1^3$. It is difficult to describe the aspect between the two vectors (apart for that both vectors are of time-like) unlike in the $\mathbb{E}^3$ spaced, it is an important point. To this case, we can not comment on the slope between the usual vector field $n(s)$ and U [7].

Theorem 3.3.

Let α be a null quaternionic curve in $\mathbb{E}_1^3$, in that case α is slant helix only if $\frac{k}{\tau}$ is fixed.

Proof. Let α be a slant helix in $\mathbb{E}_1^3$, $< n(s), U >$ is fixed. According to description we have, α is a slant helix. So

$$< n(s), U >= cons.$$

If we differentiate the equations both sides, then we have

$$< n'(s), U > + < n(s), U' >= 0.$$

Since the derivative of the constant is zero, then

$$< n'(s), U >= 0.$$

By using equations $n' = \tau u$, the following expression is obtained;

$$< \tau u, U >= 0$$
$$\tau < u, U >= 0, \tau \neq 0$$
$$< u, U >= 0.$$

Here, if we differentiate the equations both sides, by then we have
$$< u', U > + < u, U' >= 0.$$
Since derivative of the constant is zero, then;

$$< u', U >= 0.$$

By using equations $u' = -\tau l - kn$ and $U = \cos\theta n + \sin\theta u$

$$< -\tau l - kn, \cos\theta n + \sin\theta u >= 0$$
$$< -\tau l, \cos\theta n > + < -\tau l, \sin\theta u > + < -kn, \cos\theta n > + < -kn, \sin\theta u >= 0$$
$$-\tau\cos\theta < l, n > -\tau\sin\theta < l, u > -k\cos\theta < n, n > -k\sin\theta < n, u >= 0.$$

Here, since $< l, n >=< l, u >=< n, u >= 0$ and $< n, n >= 1$, the following expression is obtained;

$$-k\cos\theta = 0$$
$$k\cos\theta = 0.$$

Case i) $k = 0 \Rightarrow \cos\theta \neq 0$ or
Case ii) $k \neq 0 \Rightarrow \cos\theta = 0.$

Theorem 3.4.

Let α be a null quaternionic curve in $\mathbb{E}_1^3$, α is a slant helix only if

$$\det(u', u'', u''') = (-\tau'' + 2k\tau^2).(2k^2\tau + (-k)'').(-3\tau' k - 2k'\tau + (-k)''\tau).$$

Proof. Suppose that $\frac{k}{\tau}$ be constant. From $u' = -\tau l - kn$, we have

$$u' = -\tau l - kn$$
$$u'' = (-\tau)' l + (-\tau).l' + (-k)' n + (-k).n'.$$

Here, if

$l' = k.u$, $n' = \tau u$ and $u' = -\tau l - kn$ equals are used

$$u'' = (-\tau)' l + (-\tau) k u + (-k)' n - k\tau u$$

$$u'' = -\tau' l - 2k\tau u + (-k)' n$$

$$\begin{aligned}
u''' &= (-\tau)' l + (-\tau)' l' - 2[(k\tau)' u + (k\tau) u'] + (-k)'' n + (-k)' n' \\
&= (-\tau)' l + (-\tau)' k u - 2[k'\tau u + k\tau' u + k\tau(-\tau l - kn\ ] + (-k)'' n + (-k)'(\tau u) \\
&= (-\tau)'' l + (-\tau)' k u - 2k'\tau u - 2k\tau' u + 2k\tau^2 l + 2k^2\tau n + (-k)'' n + (-k)''\tau u \\
&= (-\tau'' + 2k\tau^2) l + (\ 2k^2\tau + (-k'')) n + ((-\tau)' k - 2k'\tau - 2k\tau' + (-k)''\tau) u \\
&= \ (-\tau'' + 2k\tau^2) l + (\ 2k^2\tau + (-k'')) n + (-3\tau' k - 2k'\tau + (-k)''\tau) u.
\end{aligned}$$

Thus, the following equation is obtained

$$\begin{aligned}
\det(u', u'', u''') &= \begin{vmatrix} -\tau'' + 2k\tau^2 & 0 & 0 \\ 0 & 2k^2\tau\ + (-k'') & 0 \\ 0 & 0 & -3\tau' k - 2k'\tau + (-k)''\tau \end{vmatrix} \\
&= (-\tau'' + 2k\tau^2)(2k^2\tau\ + (-k''))(-3\tau' k - 2k'\tau + (-k)''\tau)\ .
\end{aligned}$$

Theorem 3.5.

Let γ be a null quaternionic curve in $\mathbb{E}_1^3, \gamma$ is cylindiricial helix $\Leftrightarrow \frac{k}{\tau}$ is constant.

Proof. Let γ be a null quaternionic curve in $\mathbb{E}_1^3$, $\frac{\tau}{k} = c$ constant and

$$\alpha = \frac{c}{\sqrt{1+c^2}} l + \frac{1}{\sqrt{1+c^2}} u.$$

$$\alpha' = \frac{c}{\sqrt{1+c^2}} l' + \frac{1}{\sqrt{1+c^2}} u'$$

Here if $l' = k.u$ and $u' = -\tau l - kn$ equals are used, we get

$$\alpha' = \frac{c}{\sqrt{1+c^2}} ku + \frac{1}{\sqrt{1+c^2}} (-\tau l - kn) .$$

Here, since $\theta \in (0, \pi), \cot\theta = c$

$$\frac{1}{sin^2\theta} = \frac{sin^2\theta + cos^2\theta}{sin^2\theta} = \frac{sin^2\theta}{cos^2\theta} + \frac{cos^2\theta}{sin^2\theta} = 1 + cot^2\theta = 1 + c^2.$$

Also

$$\frac{1}{sin^2\theta} = 1 + c^2$$

$$sin^2\theta = \frac{1}{1+c^2}$$

and

$$\cos\theta = \sin\theta \cot\theta$$
$$= \frac{1}{\sqrt{1+c^2}} . c$$
$$\cos\theta = \frac{c}{\sqrt{1+c^2}} .$$

Thus,

$$\sin\theta = \frac{1}{1+c^2}$$
$$\cos\theta = \frac{c}{\sqrt{1+c^2}}$$
$$< l, \alpha > = < l, \frac{c}{\sqrt{1+c^2}} l + \frac{1}{\sqrt{1+c^2}} n >$$
$$= < l, l > \frac{c}{\sqrt{1+c^2}} + < l, n > \frac{1}{\sqrt{1+c^2}}$$
$$< l, \alpha > = \frac{c}{\sqrt{1+c^2}} = \cos\theta,$$

where θ is the constant bevel between the vectors $l(t)$ and the vector α.
Let α be a constant unit vector and $\theta \in (0, \pi), < l, \alpha > = \cos\theta$, then we can write α vector as follows:

$$\alpha = < l, \alpha > l + < n, \alpha > n + < u, \alpha > u.$$

Since

$$k. < n, \alpha > = < kn, \alpha > = < l', \alpha > = 0 \ and \ k \neq 0$$
$$< n, \alpha > = 0.$$

Also

$$\alpha = \frac{c}{\sqrt{1+c^2}} l + \frac{1}{\sqrt{1+c^2}} u$$

$$< \alpha, u >= \frac{c}{\sqrt{1+c^2}} < l, u > + \frac{1}{\sqrt{1+c^2}} < u, u > .$$

Here, since $< l, n >=< l, u >=< n, u >= 0$ and $< n, n >= 1$, the following expression is obtained

$$< \alpha, u > = \frac{1}{\sqrt{1+c^2}} = \sin\theta$$
$$< \alpha, u > = \sin\theta,$$

and

$$\alpha = \cos\theta . l + < \alpha, u > u.$$

Since α is constant unit vector,

$$\parallel \alpha \parallel = \sqrt{\left(\frac{c}{\sqrt{1+c^2}}\right)^2 + \left(\frac{1}{\sqrt{1+c^2}}\right)^2}$$

$$= \sqrt{\frac{c^2}{1+c^2} + \frac{1}{1+c^2}}$$

$$= \sqrt{\frac{c^2+1}{1+c^2}} = 1$$

$$\parallel \alpha \parallel^2 = 1 = cos^2\theta + sin^2\theta.$$

Also, because $< \alpha, u >= \sin\theta \Rightarrow < \alpha, u >^2 = sin^2\theta$ and α is constant unit vector,

$$< u, \alpha > = \pm \sin\theta$$
$$\alpha = \cos\theta . l \pm \sin\theta . u = 0$$
$$\alpha' = \cos\theta . l' \pm \sin\theta . u' = 0.$$

Here, $l' = k.u$ and $u' = -\tau l - kn$ equals are used

$$\alpha' = \cos\theta . ku \pm \sin\theta . (-\tau l - kn).$$

Also

$$< \alpha', l >= \cos\theta . k < u, l > + \sin\theta . (-\tau) < l, l > + \sin\theta . (-k) < n, l > = 0$$
$$< \alpha', n > = \cos\theta . k < u, n > + \sin\theta . (-\tau) < l, n > + \sin\theta . (-k) < n, n > = 0$$
$$< \alpha', u > = \cos\theta . k < u, u > + \sin\theta . (-\tau) < l, u > + \sin\theta . (-k) < n, u > = 0.$$

Here, since $< l, n > =< l, u > =< n, u > = 0$ and $< l, l >= < n, n > =< u, u > = 1$, the following expression is obtained

$$\sin\theta . (-\tau) + \sin\theta . (-k) + \cos\theta . k = 0$$
$$\cos\theta . k = \sin\theta . \tau + \sin\theta . k$$
$$\cos\theta . k = \sin\theta . (\tau + k)$$
$$\frac{\cos\theta}{\sin\theta} = \frac{(\tau+k)}{k}$$
$$\cot\theta = \frac{\tau}{k} + 1$$
$$\frac{\tau}{k} = \cot\theta - 1$$

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