Commun. Fac. Sci. Univ. Ank. Series A1
V. 47. pp. 139-149 (1998)

CLERK MAXWELL THEORY FOR ULTRAHYPERBOLIC OPERATORS

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(Received Aug. 4, 1998; Accepted Sep. 21, 1998)

ABSTRACT

In this study, some properties of Clerk Maxwell Theory related to Laplace operator and homogeneous polynomials are expended for ultrahyperbolic and homogeneous polynomials.

1. INTRODUCTION

In the well known Clerk Maxwell Theory related to spherical harmonics [5; p. 212] it is shown that if $f_n(x,y,z)$ is a homogeneous polynomial of degree n, then

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\frac{1}{r}=(-1)^n\frac{(2n)!}{2^n n!}\frac{1}{r^{2n+1}}\left[1-\frac{r^2\nabla^2}{2.(2n-1)}+\frac{r^4\nabla^4}{2.4.(2n-1).(2n-3)}-\dots\right]f_n(x,y,z)$$

where $r^2 = x^2 + y^2 + z^2$ and $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

In this study, Clerk Maxwell Theory is expended for the ultrahyperbolic operator defined by

$$Lu=\sum_{i=1}^{p}\frac{\partial^2 u}{\partial x_i^2}-\sum_{j=1}^{q}\frac{\partial^2 u}{\partial y_j^2} \tag{1}$$

and the Lorentzian distance r defined by $r^2=\sum_{i=1}^{p}x_i^2-\sum_{j=1}^{q}y_j^2$. The domain of L is the set of real valued function u(x,y) in $C^2(D)$, where $x = (x_1,\dots,x_p)$ and $y = (y_1,\dots,y_q)$ are the points in R^p and R^q and respectively and D is the domain of u in R^{p+q}.

2. SOME LEMMAS

In this section, we give some lemmas which are extensions of some theorems given in Maxwell Theory, to the p+q dimensional space.

Lemma 1. Let $f_n(x,y)$ and $\psi_n(x,y)$ be any homogeneous polynomial of degree n such that they are homogeneous separately of degree k and s, (k+s = n) of the variables $x = (x_1,...,x_p)$ and $y = (y_1,...,y_q)$ respectively. Then

$$f_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\ \psi_n(x,y) = \psi_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\ f_n(x,y) \tag{2}$$

where $\frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_p})$ and $\frac{\partial}{\partial y} = (\frac{\partial}{\partial y_1},...,\frac{\partial}{\partial y_q})$.

Proof. Let n be a natural number, A_i, B_j be real constants and let $K = \{1,2,...,N\}$ be a set of integers, such that N is the number of the maximum term in a n-th degree homogeneous polynomial of p+q variables. Then the functions f_n and ψ_n explicitly can be written as

$$f_n(x,y) = \sum_{i\in K} A_i x_1^{k_1^i}...x_p^{k_p^i} y_1^{s_1^i}...y_q^{s_q^i}\ ,\ \sum_{v=1}^{p} k_v^i = k\ ,\ \sum_{v=1}^{q} s_v^i = s \tag{3}$$

$$\psi_n(x,y) = \sum_{j\in K} B_j x_1^{k_1^j}...x_p^{k_p^j} y_1^{s_1^j}...y_q^{s_q^j}\ ,\ \sum_{v=1}^{p} k_v^j = k\ ,\ \sum_{v=1}^{q} s_v^j = s \tag{4}$$

where k_v^i, k_v^j, s_v^i, s_v^j are elements of the set $\{0,1,...,n\}$ and $k + s = n$. In view of (3) and (4), the left hand side of (2) can be rewritten as

$$f_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\ \psi_n(x,y) = \left\{\sum_{i\in K} A_i\left(\frac{\partial}{\partial x_1}\right)^{k_1^i}...\left(\frac{\partial}{\partial x_p}\right)^{k_p^i}\left(\frac{\partial}{\partial y_1}\right)^{s_1^i}...\left(\frac{\partial}{\partial y_q}\right)^{s_q^i}\right\}$$
$$.\left\{\sum_{j\in K} B_j x_1^{k_1^j}...x_p^{k_p^j} y_1^{s_1^j}...y_q^{s_q^j}\right\}$$

From the right hand side of this final equality, we can see that the terms for which $i \neq j$ are diminished, since for these terms the order of the derivatives are higher than the power of the variables and $\sum_{v=1}^{p} k_v^i = \sum_{v=1}^{p} k_v^j = k$ and $\sum_{v=1}^{q} s_v^i = \sum_{v=1}^{q} s_v^j = s$. Thus

$$f_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\ \psi_n(x,y) = \sum_{i\in K} A_i B_i (k_1^i)!...(k_p^i)!.(s_1^i)!...(s_q^i)! \tag{5}$$

Similarly, using of (3) and (4), the right hand side of (2) can be rewritten as

$$\begin{aligned}\psi_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) f_n(x,y) &= \left\{\sum_{j\in K} B_j\left(\frac{\partial}{\partial x_1}\right)^{k_1^j}\cdots\left(\frac{\partial}{\partial x_p}\right)^{k_p^j}\left(\frac{\partial}{\partial y_1}\right)^{s_1^j}\cdots\left(\frac{\partial}{\partial y_q}\right)^{s_q^j}\right\}\\ &\cdot\left\{\sum_{i\in K} A_i x_1^{k_1^i}\cdots x_p^{k_p^i} y_1^{s_1^i}\cdots y_q^{s_q^i}\right\}\\ &= \sum_{i\in K} A_i B_i (k_1^i)!\ldots(k_p^i)!.(s_1^i)!\ldots(s_q^i)!\end{aligned} \tag{6}$$

Thus, (5) and (6) complete the proof.

Remark. For the later discussion, here we use the homogeneity with respect to the variables x and y, but this lemma can be proved without homogeneity assumption separately on the variables x and y.

Lemma 2. Let $\psi_n(x,y)$ be any homogeneous polynomial of degree n, such that it is homogeneous separately of degree k and s, (k+s = n) of the variables $x = (x_1,\ldots,x_p)$ and $y = (y_1,\ldots,y_q)$ respectively. Then

$$\begin{aligned}&\left(x_1\frac{\partial}{\partial u_1}+\ldots+x_p\frac{\partial}{\partial u_p}-y_1\frac{\partial}{\partial v_1}-\ldots-y_q\frac{\partial}{\partial v_q}\right)^n \psi_n(u_1,\ldots,u_p,v_1,\ldots,v_q)\\ &= (-1)^s\, n!\; \psi_n(x_1,\ldots,x_p,y_1,\ldots,y_q)\end{aligned} \tag{7}$$

Proof. Let n be a natural number and let K be a set of integers, $K = \{1,2,\ldots,N\}$, such that N is the number of maximum term in a n-th degree homogeneous polynomial of p+q variables. For $k_1^i+\ldots+k_p^i+s_1^i+\ldots+s_q^i = n$, using the binomial theorem [1,p.823], we have

$$\begin{aligned}&\left(x_1\frac{\partial}{\partial u_1}+\ldots+x_p\frac{\partial}{\partial u_p}-y_1\frac{\partial}{\partial v_1}-\ldots-y_q\frac{\partial}{\partial v_q}\right)^n = (-1)^s\sum_{i\in K}\frac{n!}{(k_1^i)!\ldots(k_p^i)!(s_1^i)!\ldots(s_q^i)!}\\ &\cdot\left(x_1\frac{\partial}{\partial u_1}\right)^{k_1^i}\cdots\left(x_p\frac{\partial}{\partial u_p}\right)^{k_p^i}\left(y_1\frac{\partial}{\partial v_1}\right)^{s_1^i}\cdots\left(y_q\frac{\partial}{\partial v_q}\right)^{s_q^i}\end{aligned} \tag{8}$$

Since $\psi_n(u,v)$ is a homogeneous polynomial of degree n, clearly

$$\psi_n(x,y) = \sum_{j\in K} B_j u_1^{k_1^j}\cdots u_p^{k_p^j} v_1^{s_1^j}\cdots v_q^{s_q^j}\;,\; \sum_{v=1}^{p} k_v^j = k\;,\; \sum_{v=1}^{q} s_v^j = s\;,\; k+s = n. \tag{9}$$

Thus by (8) and (9), we have

$$\left(x_1 \frac{\partial}{\partial u_1} + \dots + x_p \frac{\partial}{\partial u_p} - y_1 \frac{\partial}{\partial v_1} - \dots - y_q \frac{\partial}{\partial v_q}\right)^n = \psi_n(u,v)$$

$$= (-1)^s \sum_{i\in K} \frac{n!}{(k_1^i)!\dots(k_p^i)!(s_1^i)!\dots(s_q^i)!} x_1^{k_1^i}\dots x_p^{k_p^i} y_1^{s_1^i}\dots y_q^{s_q^i}$$

$$\cdot \left(\frac{\partial}{\partial u_1}\right)^{k_1^i} \dots \left(\frac{\partial}{\partial u_p}\right)^{k_p^i} \left(\frac{\partial}{\partial v_1}\right)^{s_1^i} \dots \left(\frac{\partial}{\partial v_q}\right)^{s_q^i} \sum_{j\in K} B_j u_1^{k_1^j}\dots u_p^{k_p^j} v_1^{s_1^j}\dots v_q^{s_q^j}$$

$$= (-1)^s n! \sum_{\substack{i\in K\\ i\neq j}} \sum_{j\in K} \frac{B_j}{(k_1^i)!\dots(k_p^i)!(s_1^i)!\dots(s_q^i)!} x_1^{k_1^i}\dots x_p^{k_p^i} y_1^{s_1^i}\dots y_q^{s_q^i}$$

$$\cdot \left(\frac{\partial}{\partial u_1}\right)^{k_1^i} \dots \left(\frac{\partial}{\partial u_p}\right)^{k_p^i} \left(\frac{\partial}{\partial v_1}\right)^{s_1^i} \dots \left(\frac{\partial}{\partial v_q}\right)^{s_q^i} u_1^{k_1^j}\dots u_p^{k_p^j} v_1^{s_1^j}\dots v_q^{s_q^j}$$

$$+ (-1)^s n! \sum_{j\in K} B_j \frac{x_1^{k_1^j}\dots x_p^{k_p^j} y_1^{s_1^j}\dots y_q^{s_q^j}}{(k_1^i)!\dots(k_p^i)!(s_1^i)!\dots(s_q^i)!} \cdot \left(\frac{\partial}{\partial u_1}\right)^{k_1^j} \dots \left(\frac{\partial}{\partial u_p}\right)^{k_p^j} \left(\frac{\partial}{\partial v_1}\right)^{s_1^j} \dots \left(\frac{\partial}{\partial v_q}\right)^{s_q^j}$$

$$\cdot u_1^{k_1^j}\dots u_p^{k_p^j} v_1^{s_1^j}\dots v_q^{s_q^j}$$

$$= 0 + (-1)^s n! \sum_{j\in K} B_j x_1^{k_1^j}\dots x_p^{k_p^j} y_1^{s_1^j}\dots y_q^{s_q^j} = (-1)^s n! \psi_n(x_1\dots x_p y_1\dots y_q)$$

which completes the proof.

Lemma 3. Let $f_n(x,y)$ be a homogeneous polynomial satisfying the assumptions of Lemma 1. In addition, let $w = \phi(x_1,\dots,x_p,y_1,\dots,y_q)$ and $F = F(w)$ be any functions having n-th order continuous derivatives respectively in a domain $D \subset R^{p+q}$ and in $\phi(D) \subset R$. Then,

$$f_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\ F(w) = \sum_{m=0}^{n-1} \chi_m \frac{d^{n-m}F(w)}{dw^{n-m}} \tag{10}$$

Here $\chi_0,\chi_1,\dots,\chi_{n-1}$ which depend only on the variables $x_1,\dots,x_p,y_1,\dots,y_q$ for which the functions to be determined.

Proof. By (3), we have

$$f_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\ F(w) = \sum_{i\in K} A_i \left(\frac{\partial}{\partial x_1}\right)^{k_1^i} \dots \left(\frac{\partial}{\partial x_p}\right)^{k_p^i} \left(\frac{\partial}{\partial y_1}\right)^{s_1^i} \dots \left(\frac{\partial}{\partial y_q}\right)^{s_q^i} F(w) \tag{11}$$

Since $w = \phi(x_1,...,x_p,y_1,...,y_q)$ and $(k_1^i + ... + k_p^i) + (s_1^i + ... + s_q^i) = k+s = n$ successive differentiation of $F(w)$ with respect to x_1 gives

$$\frac{\partial F(w)}{\partial x_1} = \frac{dF}{dw} \cdot \frac{\partial w}{\partial x_1}$$

$$\frac{\partial^2 F(w)}{\partial x_1^2} = \frac{d^2 F}{dw^2} \cdot \left(\frac{\partial w}{\partial x_1}\right)^2 + \frac{dF}{dw} \cdot \frac{\partial^2 w}{\partial x_1^2}$$

....

$$\frac{\partial^{k_1^i} F(w)}{\partial x_1^{k_1^i}} = B_0^i \frac{d^{k_1^i} F}{dw^{k_1^i}} + B_1^i \frac{d^{k_1^i-1} F}{dw^{k_1^i-1}} + ... + B_{k_1^i-1}^i \frac{dF}{dw}$$

where the coefficients $B_0^i, B_1^i, ..., B_{k_1^i-1}^i$ are the functions $x_1, ..., x_p, y_1, ..., y_q$. Similarly, taking the required derivatives with respect to the remaining variables, we get

$$\left(\frac{\partial}{\partial x_1}\right)^{k_1^i} ... \left(\frac{\partial}{\partial x_p}\right)^{k_p^i} \left(\frac{\partial}{\partial y_1}\right)^{s_1^i} ... \left(\frac{\partial}{\partial y_q}\right)^{s_q^i} F(w) = \sum_{m=0}^{n-1} P_m^i \frac{d^{n-m} F(w)}{dw^{n-m}}$$

Since P_m^i are the functions of $x_1,...,x_p,y_1,...,y_q$ by (11)

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) F(w) = \sum_{i \in K} A_i \sum_{m=0}^{n-1} P_m^i \frac{d^{n-m} F(w)}{dw^{n-m}}$$

$$= \sum_{m=0}^{n-1} \left(\sum_{i \in K} A_i P_m^i\right) \frac{d^{n-m} F(w)}{dw^{n-m}} = \sum_{m=0}^{n-1} \chi_m \frac{d^{n-m} F(w)}{dw^{n-m}}$$

where $\chi_m = \sum_{i \in K} A_i P_m^i$ which is what we needed.

Lemma 4. Let $f_n(x,y)$ and $\phi(x,y)$ be as in Lemma 3. Then

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \{\phi(x,y)\}^n = n! \sum_{m=0}^{n-1} \frac{1}{m!} \chi_m(x,y)\, \phi^m(x,y) \tag{12}$$

Proof. If we let $\phi(x,y) = w$ and $F(w) = w^n$ i.e.

$$F(w) = F(\phi(x,y)) = F\{\phi(x_1,...,x_p,y_1,...,y_q)\} = \{\phi(x_1,...,x_p,y_1,...,y_q)\}^n$$

by (10) the proof follows easily.

Lemma 5. Let f_n and ϕ be as in Lemma 3 and let $u = (u_1, ..., u_p)$ and $v = (v_1,...,v_q)$. Then

$$f_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y}) \{\phi(x,y)\}^n = \lim_{(u,v)\to(0,0)} f_n(\frac{\partial}{\partial u},\frac{\partial}{\partial v}) \{\phi(x+u,y+v)\}^n \tag{13}$$

where $\frac{\partial}{\partial u} = (\frac{\partial}{\partial u_1}, ..., \frac{\partial}{\partial u_p})$ and $\frac{\partial}{\partial v} = (\frac{\partial}{\partial v_1}, ..., \frac{\partial}{\partial v_q})$.

Proof. Let $(x,y) \in D$ be fixed point. Consider the neighborhood points $(x+u,y+v) \in D$ and replace (x,y) by $(x+u,y+v)$ in (12). Since $\frac{\partial}{\partial(x+u)} = \frac{\partial}{\partial u}$ and $\frac{\partial}{\partial(y+v)} = \frac{\partial}{\partial v}$ by (12) we can write

$$f_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y}) \{\phi(x+u,y+v)\}^n = n! \Big\{\chi_0(x+u,y+v) + \chi_1(x+u,y+v)\ \phi(x+u,y+v) + ... + \frac{1}{(n-1)!}\ \chi_{n-1}(x+u,y+v)\ \phi^{n-1}(x+u,y+v)\Big\}$$

and from this taking the limit $(u,v) \to (0,0)$, we get

$$\lim_{(u,v)\to(0,0)} f_n(\frac{\partial}{\partial u},\frac{\partial}{\partial v}) \{\phi(x+u,y+v)\}^n = n!\Big\{\chi_0(x,y) + \chi_1(x,y)\phi(x,y) + ... + \frac{1}{(n-1)!}\ \chi_{n-1}(x,y)\phi^{n-1}(x,y)\Big\} \tag{14}$$

Since the right hand sides of (12) and (14) are equal the left hand sides must be equal. That proves the equality given in (13).

3. FUNDAMENTAL THEOREMS

In this section, we give the exact expressions for the coefficients of the (10) type expansions and we investigate some relations between χ_m and ultrahyperbolic operator L.

Theorem 1. Let $f_n(x,y)$ be any homogeneous polynomial of degree n such that it is homogeneous separately of degree k and s, (k+s = n) of the variables $x = (x_1,...,x_p)$ and $y = (y_1,...,y_q)$ respectively. In addition let $w = \phi(x_1,...,x_p,y_1,...,y_q)$ and $F = F(w)$ be any functions having n-th order continuous derivatives respectively in a domain D of p+q dimensional space and in $\phi(D) \subset R$, then

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) F(\phi) = \sum_{m=0}^{n-1} \left\{ \frac{1}{(n-m)!} \lim_{(u,v)\to(0,0)} f_n\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right) \left[\phi(x+u,y+v) - \phi(x,y)\right]^{n-m} \right\} \cdot \frac{d^{n-m}F(\phi)}{d\phi^{n-m}} \tag{15}$$

Proof. To prove the theorem first let

$$\{\phi(x+u,y+v)\}^n = \{\phi(x,y) + [\phi(x+u,y+v) - \phi(x,y)]\}^n$$

By applying the binomial theorem to the right hand side of this, we get

$$\{\phi(x+u,y+v)\}^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \{\phi(x+u,y+v) - \phi(x,y)\}^{n-m} \{\phi(x,y)\}^m$$

Using this result in (13), we obtain

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) \{\phi(x,y)\}^n = \lim_{(u,v)\to(0,0)} f_n\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right) \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \cdot \{\phi(x+u,y+v) - \phi(x,y)\}^{n-m} \{\phi(x,y)\}^m$$

Since the last term drops out from the right hand side of this equality, we get

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) \{\phi(x,y)\}^n = n! \sum_{m=0}^{n-1} \frac{n!}{m!(n-m)!} \cdot \lim_{(u,v)\to(0,0)} f_n\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right) \{\phi(x+u,y+v) - \phi(x,y)\}^{n-m} \{\phi(x,y)\}^m \tag{16}$$

Comparing (12) and (16), we obtain

$$\chi_m(x,y) = \frac{1}{(n-m)!} \lim_{(u,v)\to(0,0)} f_n\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right) \{\phi(x+u,y+v) - \phi(x,y)\}^{n-m} \tag{17}$$

where $m = 0,1,\ldots,n-1$.

On the other hand, with $w = \phi$ in (10) we have

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) F(\phi) = \sum_{m=0}^{n-1} \chi_m(x,y) \frac{d^{n-m}F(\phi)}{d\phi^{n-m}} \tag{18}$$

By (17) and (18), we have the result.

Theorem 2. Let f_n and F be functions as in Theorem 1. If L is the ultrahyperbolic operator defined in (1) and

$$\phi(x,y) = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j^2 ,$$

then

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) F(\phi) = (-1)^s \sum_{m=0}^{[n/2]} \frac{2^{n-2m}}{m!} \frac{d^{n-m}F(\phi)}{d\phi^{n-m}} L^m f_n(x,y) \tag{19}$$

Proof. By the above definition of $\phi(x,y)$

$$[\phi(x+u,y+v) - \phi(x,y)]^{n-m}$$
$$= \left\{(x_1+u_1)^2+...+(x_p+u_p)^2 - \left[(y_1+v_1)^2+...+(y_q+v_q)^2\right] - \left[x_1^2+...+x_p^2 - (y_1^2+...+y_q^2)\right]\right\}^{n-m}$$
$$= \left[2(x_1u_1 + ... + x_pu_p - y_1v_1 - ... - y_qv_q) + (u_1^2 + ... + u_p^2 - v_1^2 - ... - v_q^2)\right]^{n-m}$$

Thus, by applying binomial expansion to the last expression above, we get

$$[\phi(x+u,y+v) - \phi(x,y)]^{n-m}$$
$$= \sum_{\mu=0}^{n-m} \binom{n-m}{\mu} \left\{2(x_1u_1 + ... + x_pu_p - y_1v_1 - ... - y_qv_q)\right\}^{n-m-\mu} (u_1^2 + ... + u_p^2 - v_1^2 - ... - v_q^2)^{\mu}$$

By substituting this in (17), and noting that all terms for which $\mu \neq m$ vanish, we have

$$\chi_m(x,y) = \frac{1}{(n-m)!} \lim_{(u,v)\to(0,0)} f_n\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right) \cdot \binom{n-m}{\mu}$$
$$\cdot \left\{2(x_1u_1 + ... + x_pu_p - y_1v_1 - ... - y_qv_q)\right\}^{n-2m} (u_1^2 + ... + u_p^2 - v_1^2 - ... - v_q^2)^{m}$$
$$= \frac{2^{n-2m}}{m!(n-2m)!} \lim_{(u,v)\to(0,0)} f_n\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right)$$
$$\cdot \left\{(x_1u_1 + ... + x_pu_p - y_1v_1 - ... - y_qv_q)\right\}^{n-2m} (u_1^2 + ... + u_p^2 - v_1^2 - ... - v_q^2)^{m} \tag{20}$$

Using Lemma 1, this equality can be written as:

$$\chi_m(x,y) = \frac{2^{n-2m}}{m!(n-2m)!} \lim_{(u,v)\to(0,0)} \left(x_1\frac{\partial}{\partial u_1} + ... + x_p\frac{\partial}{\partial u_p} - y_1\frac{\partial}{\partial v_1} - ... - y_q\frac{\partial}{\partial v_q}\right)^{n-2m}$$
$$\left(\frac{\partial^2}{\partial u_1^2} + ... + \frac{\partial^2}{\partial u_p^2} - \frac{\partial^2}{\partial v_1^2} - ... - \frac{\partial^2}{\partial v_q^2}\right)^m f_n(u_1,...,u_p,v_1,...,v_q) \tag{21}$$

Let

$$\left(\frac{\partial^2}{\partial u_1^2}+\dots+\frac{\partial^2}{\partial u_p^2}-\frac{\partial^2}{\partial v_1^2}-\dots-\frac{\partial^2}{\partial v_q^2}\right)^m f_n(u_1,\dots,u_p,v_1,\dots,v_q)$$

$$= g_{n-2m}(u_1,\dots,u_p,v_1,\dots,v_q) \tag{22}$$

Thus, g_{n-2m} is a homogeneous polynomial of degree n–2m and so by Lemma 2,

$$\left(x_1\frac{\partial}{\partial u_1}+\dots+x_p\frac{\partial}{\partial u_p}-y_1\frac{\partial}{\partial v_1}-\dots-y_q\frac{\partial}{\partial v_q}\right)^{n-2m} g_{n-2m}(u_1,\dots,u_p,v_1,\dots,v_q)$$

$$= (-1)^s(n-2m)!\,g_{n-2m}(x_1,\dots,x_p,y_1,\dots,y_q) \tag{23}$$

Hence by (21), (22) and (23) we can write

$$\chi_m(x,y) = (-1)^s \frac{2^{n-2m}}{m!}\, g_{n-2m}(x_1,\dots,x_p,y_1,\dots,y_q)$$

On the other hand, considering (22) and (1), we get

$$\chi_m(x,y) = (-1)^s \frac{2^{n-2m}}{m!}\left(\frac{\partial^2}{\partial x_1^2}+\dots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial y_1^2}-\dots-\frac{\partial^2}{\partial y_q^2}\right)^m f_n(x_1,\dots,x_p,y_1,\dots,y_q)$$

$$= (-1)^s \frac{2^{n-2m}}{m!}\, L^m\, f_n(x_1,\dots,x_p,y_1,\dots,y_q) \tag{24}$$

Hence by substituting the values of χ_m, m = 0,1,2,...,n–1 of (24) in (18), we obtain (19). We remark that the number of the terms in (18) is n, but the number of the terms in (19) is less than n. This is because

$$L_n^m\, f_n(x,y) = 0 \text{ for } \left[\frac{n}{2}\right] < m \le n-1$$

where

$$\left[\frac{n}{2}\right] = \begin{cases} \dfrac{n}{2} & ; \text{ if n is even} \\[2mm] \dfrac{n-1}{2} & ; \text{ if n is odd} \end{cases}$$

Theorem 3. Let $f_n(x,y)$ be as in Theorem 1. Then

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) r^{2-p-q}$$

$$= (-1)^k \sum_{m=0}^{[n/2]} (-1)^m \frac{1}{m!2^m} \frac{(p+q-2)(p+q)(p+q+2)\ldots(p+q+2(n-m)-4)}{r^{p+q+2(n-m)-2}} \cdot L^m f_n(x,y) \quad (25)$$

where L is the ultrahyperbolic operator defined in (1), and r is the Lorentzian distance defined by

$$r^2 = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j^2 = |x|^2 - |y|^2 > 0 \quad (26)$$

Proof. In (19), let $\phi = r^2$ and $F(\phi) = \phi^{\frac{2-p-q}{2}} = r^{2-p-q}$. Then, since

$$\frac{d^nF(\phi)}{d\phi^n} = \frac{d^n(r^{2-p-q})}{d(r^2)^n} = (-1)^n \frac{(p+q-2)(p+q)\ldots(p+q+2n-4)}{2^n} r^{2-p-q-2n}$$

and substituting this expression of $\frac{d^nF(\phi)}{d\phi^n}$ in the right hand side of (19) with n replaced by n–m we obtain (25).

Conclusion 1. Let $f_n(x,y)$ be as in Theorem 1. If f_n is a solution of the ultrahyperbolic equation $Lu = 0$, then

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) F(\phi) = (-1)^s 2^n \frac{d^nF(\phi)}{d\phi^n} f_n(x,y) \quad (27)$$

and

$$f_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) r^{2-p-q} = (-1)^k \frac{(p+q-2)(p+q)\ldots(p+q+2n-4)}{r^{p+q+2n-2}} f_n(x,y) . \quad (28)$$

Proof. Since $f_n(x,y)$ is a solution of the equation $Lu = 0$, $Lf_n = L^2f_n = \ldots = L^mf_n = 0$. Using this facts in (19) and (25) we obtain respectively (27) and (28).

Theorem 4. Let $f_n(x,y)$ be a homogeneous polynomial defined in Theorem 1 and let r be given as in (26). Then the expression

$$\sum_{m=0}^{[n/2]} (-1)^m \frac{1}{m!2^m} \frac{(p+q-2)(p+q)(p+q+2)\ldots(p+q+2(n-m)-4)}{r^{p+q+2(n-m)-2}} L^m f_n(x,y) \quad (29)$$

is a solution of the equation $Lu = 0$.

Proof. By using the definitions of L and r, it is easy to show that

$$L(r^{\alpha}) = \alpha(\alpha + p + q - 2)r^{\alpha-2}$$

for any real parameter α. Thus $L(r^{2-p-q}) = 0$. On the other hand, since both $f_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y})$ and L operators have constant coefficients, $Lf_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y}) = f_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y})L$ has the commutative property. Therefore,

$$L\left[f_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\ r^{2-p-q}\right] = f_n(\frac{\partial}{\partial x},\frac{\partial}{\partial y})L(r^{2-p-q}) = 0$$

This shows that, the left hand side of (25) and thus the right hand side of (25) which is the same as the left hand side, that is the expression in (29) is a solution of the equation Lu = 0. This completes the proof.

Note that, since Lu = 0 is a linear homogeneous equation, it is obvious that the multiplier $(-1)^k$ appeared on the right hand side of (25) can be omitted.

Remark. This theorem says that for any homogeneous polynomial $f_n(x,y)$ can be used to obtain a solution to Lu = 0 so that $f_n(x,y)$ is not a solution of the equation Lu = 0 but it satisfies the conditions of conclusion 1.

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