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ON THE INEQUIVALENCE AND STANDARD BASIS OF THE SPECHT MODULES OF THE HYPEROCTAHEDRAL GROUPS

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ABSTRACT

The representations of the hyperoctahedral groups O_n has been studied by many authors, see for example Al-Aamily, Morris and Peel and Morris. The latter author has interpreted the work of the first three authors in the combinatorial language used in the representation theory of the symmetric groups, but a work on the inequivalence and standard basis of the Specht modules of O_n has not yet appeared in the literature. Therefore, in this paper we show that Specht modules of the hyperoctahedral groups are mutually non-isomorphic and determine the standard basis of the Specht modules.

1. INTRODUCTION

An account of the irreducible representations of the hyperoctahedral groups O_n can be found in a variety of places, see for example Mayer [3], Al-Aamily, Morris and Peel [1] and Morris [4]. The first author has constructed simple left ideals in the classical case for these groups. The last author has translated the work of [1] to the language of the combinatorial concepts used in the representation theory of the symmetric groups. We first establish the basic notation and state some results which are required later. We refer the reader to [1] and [4] for much of the undefined terminology.

The hyperoctahedral group O_n (or C_n in the notation of Weyl groups) is the group of all permutations σ of $\{\pm1, \pm2, ..., \pm n\}$ such that $\sigma(-i) = -\sigma(i)$ for $i = 1, 2, ..., n$. A positive transposition has the form (a, b)(-a, -b). These generate a subgroup of O_n isomophic to the symmetric group S_n. A negative transposition is of the form (a, -a). These generate a normal subgroup isomorphic to $C_2 \times C_2 \times ... \times C_2$ (n factors).

A pair of partitions of n (λ, μ) consists of partitions $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ and $\mu = (\mu_1, \mu_2, ..., \mu_s)$ such that λ is a partition of $|\lambda|$ and μ is a partition of $|\mu|$, and $|\lambda| + |\mu| = n$. A double Young diagram [,] is defined in the obvious way. A partial order on the set of pairs of partitions of n may be defined as follows. Let (λ, μ) and (λ', μ') be pairs of partitions of n. (λ, μ) dominates (λ', μ'), written $(\lambda, \mu) \trianglerighteq (\lambda', \mu')$ if $|\lambda| > |\lambda'|$ or if $|\lambda| = |\lambda'|$, $|\mu| = |\mu'|$ with $\lambda \trianglerighteq \lambda'$ and $\mu \trianglerighteq \mu'$.

Let (λ, μ) be a pair of partitions of n. A (λ, μ)-tableau t is an array of integers obtained by replacing each node in $[\lambda, \mu]$ by one of the integers $\pm 1, \pm 2, ..., \pm n$ with i and -i (i = 1. ..., n) not appearing simutaneously. A (λ, μ)-tableau t will be sometimes written (t_λ, t_μ). Let

$$t(i, j, k) = \begin{cases} t_\lambda(i, j) & \text{if } k = 1 \\ t_\mu(i, j) & \text{if } k = 2, \end{cases}$$

where $t_\lambda(i, j)$ and $t_\mu(i, j)$ stand for the entry of t_λ and t_μ in in position (i, j), as in a matrix, respectively. Let R_t (resp. C_t) be the group of row (resp. column) permutations of t.

Two, (λ, μ)-tableaux t_1 and t_2 are row equivalent, written $t_1 \sim t_2$, if there exists $\sigma \in R_{t_1}$ such that $t_2 = \sigma t_1$. The equivalence class which contains the (λ, μ)-tableau t is $\{t\}$ and is called a (λ, μ)-tabloid. The Group O_n acts on the set of (λ, μ)-tabloids by $\sigma\{t\} = \{\sigma t\}$ for all $\sigma \in O_n$.

Let K be an arbitrary field. If (λ, μ) is a pair of partitions of n, let $M^{\lambda,\mu}$ be the vector space over K spanned by (λ, μ)-tabloids. By extending the above action linearly to KO_n, we have that $M^{\lambda,\mu}$ is a KO_n-module. As in the case of the symmetric group (see [2]), a bilinear form $\langle.,.\rangle$ on $M^{\lambda,\mu}$ is defined in the obvious way.

Let t be a (λ, μ)-tableau. Define $\kappa_t \in KO_n$ by $\kappa_t = \sum_{\sigma \in C_t} (\text{sgn } \sigma)\sigma$, where $\text{sgn}(\sigma) = (-1)^{\ell(\sigma)}$ is the sign function and $\ell(\sigma)$ is the length of σ. The (λ, μ)-polytabloid e_t associated with the tableau t is given by $e_t = \kappa_t\{t\}$. The Specth module $S^{\lambda,\mu}$ for the pair of partitions (λ, μ) is the submodule of $M^{\lambda,\mu}$ spanned by (λ, μ)-polytabloids.

1.1. Let t be a (λ, μ)-tableau and t' be a (λ', μ')-tableau such that for all a, if a occurs in t_λ then $\pm a$ occurs in $t'_{\lambda'}$. Suppose that a, b belong to the same row of $t'_{\lambda'}$ implies that c, d belong to different columns of t_γ, where $c = \pm a$, $d = \pm b$ and $\gamma = \lambda$ or μ. Then $(\lambda, \mu) \trianglerighteq (\lambda', \mu')$.

1.2. Let t be a (λ, μ)-tableau and $\sigma \in O_n$. Then we have the following facts:

(i) $e_{\sigma t} = \sigma e_t$, and if $\sigma \in C_t$ then set $= (\text{sgn } \sigma)e_t$.

(ii) If $u \in M^{\lambda,\mu}$ then $\kappa_t u$ is a multiple of e_t.

(iii) Let t' be another (λ', μ')-tableau. If there exist a, b in the same row of $t'_{\gamma'}$, such that c, d are in the same column of t_γ, where $c = \pm a$, $d = \pm b$ and $\gamma = \lambda$ or μ, then $\kappa_t\{t'\} = 0$.

1.3. If U is a submodule of $M^{\lambda,\mu}$ than either $U \supseteq S^{\lambda,\mu}$ or $U \subseteq (S^{\lambda,\mu})^\perp$, where $(S^{\lambda,\mu})^\perp$ is the complement of $S^{\lambda,\mu}$ in $M^{\lambda,\mu}$. Furthermore, the module $S^{\lambda,\mu}/S^{\lambda,\mu} \cap (S^{\lambda,\mu})^\perp$ is zero or irreducible.

2. INEQUIVALENCE

In this section, we prove that Specht modules of O_n are mutually non-isomorphic.

Lemma 2.1. Let t be a (λ, μ)-tableau. Then

(i) $\kappa_t\kappa_t = |C_t|\kappa_t$,
(ii) $\kappa_t e_t = |C_t|e_t$.

Proof. We know that $\kappa_t = \sum_{\sigma\in C_t} (\text{sgn } \sigma)\sigma$. Hence,

$$\begin{aligned}
\kappa_t\kappa_t &= \left(\sum_{\sigma\in C_t} (\text{sgn } \sigma)\sigma\right)\left(\sum_{\pi\in C_t} (\text{sgn } \pi)\pi\right) \\
&= \sum_{\sigma,\pi\in C_t} (\text{sgn } \sigma)\ (\text{sgn } \pi)\sigma\pi \\
&= \sum_{\sigma,\gamma\in C_t} (\text{sgn } \sigma)\ (\text{sgn } \sigma^{-1}\gamma)\gamma \qquad (\text{where } \sigma\pi = \gamma) \\
&= \sum_{\sigma\in C_t}\sum_{\gamma\in C_t} (\text{sgn } \gamma)\gamma = |C_t|\kappa_t, \text{ proving (i)}
\end{aligned}$$

(ii) follows from (i)

Lemma 2.2. Let t be a (λ, μ)-tableau and t' be a (λ', μ')-tableau. Suppose that $|\lambda| > |\lambda'|$. Then we have the following facts:

(i) $\kappa_t\{t'\} = 0$
(ii) $\kappa_t e_{t'} = 0$.

Proof. Since $|\lambda| > |\lambda'|$ then there exists at least one entry a in t_λ such that $\pm a$ occurs in $t'_{\lambda'}$. Then $(a, -a) \in C_t \cap R_{t'}$. Thus, $(e-(a, -a))\ \{t'\} = 0$. Since $(a, -a) \in C_t$ then $(a, -a)$ generates a subgroup of order 2 of C_t. Select signed coset representatives $\sigma_1, \ldots, \sigma_k$ for this subgroup of C_t; then

$$\kappa_t\{t'\} = \left(\sum_{i=1}^{k} \sigma_i\right) (e - (a, -a))\ \{t'\} = 0 \quad \text{as required.}$$

(ii) follows immediately from (i).

Lemma 2.3. Let t be (λ, μ)-tableau and t' a (λ', μ')-tableau such that for all a, if a occurs in t_λ then $\pm a$ occurs in $t'_{\lambda'}$. Suppose that $\kappa_t\{t'\} \neq 0$. Then $(\lambda, \mu) \trianglerighteq (\lambda', \mu')$.

Proof. Suppose that a and b are two elements in the same row of $t'_{\gamma'}$, where $\gamma' = \lambda'$ or μ'. Then c and d cannot be in the same column of t_γ, where $c = \pm a$, $d = \pm b$ and $\gamma = \lambda$ or μ, for if so, then by (1.2) (iii) $\kappa_t\{t'\} = 0$, contradicting our hypothesis. Thus, (1.1) yields $(\lambda, \mu) \trianglerighteq (\lambda', \mu')$.

Proposition 2.4. Suppose that the field of scalars is Q and $\theta \in \mathrm{Hom}(S^{\lambda,\mu}, S^{\lambda',\mu'})$ where $|\lambda| > |\lambda'|$. Then $\theta = 0$.

Proof. Let $\theta \in \mathrm{Hom}(S^{\lambda,\mu}, S^{\lambda',\mu'})$ where $|\lambda| > |\lambda'|$. Let t be a (λ, μ)-tableau and t' be a (λ', μ')-tableau. Then

$$\theta(e_t) = \sum \alpha_{t'} e_{t'},$$

where the summation is taken over all (λ', μ')-tableaux. By Lemma 2.1 and Lemma 2.2 we have

$$|C_t|\, \theta(e_t) = \theta(\kappa_t e_t) = \kappa_t \theta(e_t) = \sum \alpha_{t'} \kappa_t e_{t'} = 0.$$

Thus $\theta(e_t) = 0$ and $\theta = 0$ as required, which implies that $S^{\lambda,\mu}$ and $S^{\lambda',\mu'}$ are not isomorphic when $|\lambda| > |\lambda'|$.

Proposition 2.5. Suppose the field of scalars is **Q** and $\theta \in$ Hom $(S^{\lambda,\mu}, M^{\lambda',\mu'})$ is non-zero, where $|\lambda| > |\lambda'|$. Thus, $(\lambda, \mu) \trianglerighteq (\lambda', \mu')$ and if $(\lambda, \mu) = (\lambda', \mu')$ then θ is multiplication by a scalar.

Proof. Since $\theta \neq 0$, there is some basis vector e_t such that $\theta(e_t) \neq 0$. Because $\langle .,. \rangle$ is an inner product with rational scalars, $M^{\lambda,\mu} = S^{\lambda,\mu} \oplus (S^{\lambda,\mu})^{\perp}$. Thus, we can extend θ to an element of Hom$(M^{\lambda,\mu}, M^{\lambda',\mu'})$ by setting $\theta((S^{\lambda,\mu})^{\perp}) = 0$. So

$$0 \neq \theta\,(e_t) = \theta(\kappa_t\,\{t\}) = \kappa_t\theta\{t\} = \kappa_t\left(\sum_i \alpha_i\{t'_i\}\right)$$

where the t'_i are (λ', μ')-tableaux.

By Lemma 2.3 we have $(\lambda, \mu) \trianglerighteq (\lambda', \mu')$. In the case $(\lambda, \mu) = (\lambda', \mu')$, (1.2) (ii) yields $\theta(e_t) = \alpha e_t$ for some scalar α. So for any $\sigma \in O_n$,

$$\theta(e_{\sigma t}) = \theta(\sigma e_t) = \sigma\theta(e_t) = \sigma(\alpha e_t) = \alpha e_{\sigma t}.$$

Thus θ is multiplication by a scalar.

We can finally verify all our claims about the Specht modules.

Theorem 2.6. Let (λ, μ) be a pair of partitions of n and $K = \mathbf{Q}$. Then the $S^{\lambda,\mu}$ give a complete set of irreducible KO_n-modules.

Proof. The $S^{\lambda,\mu}$ are irreducible by (1.3) and the fact that $S^{\lambda,\mu} \cap (S^{\lambda,\mu})^{\perp} = \{0\}$ for the field **Q**. Since we have the right number of modules for a full set, it suffices to show that they are pairwise inequivalent.

Now, let (λ, μ) and (λ', μ') be pairs of partitions of n. If $|\lambda| > |\lambda'|$ then by Proposition 2.4, $S^{\lambda,\mu}$ and $S^{\lambda',\mu'}$ are not isomorphic. If $|\lambda| = |\lambda'|$ and $S^{\lambda,\mu} \cong S^{\lambda',\mu'}$ then there exists a non-zero homomorphism $\theta \in$ Hom $(S^{\lambda,\mu}, M^{\lambda',\mu'})$ since $S^{\lambda',\mu'} \subseteq M^{\lambda',\mu'}$. Thus by Proposition 2.5, $(\lambda, \mu) \trianglerighteq (\lambda', \mu')$. Similarly, $(\lambda', \mu') \trianglerighteq (\lambda, \mu)$, so that $(\lambda, \mu) = (\lambda', \mu')$.

3. THE STANDARD BASIS OF THE SPECHT MODULES

In this section, we determine the standard basis of the Specht modules $S^{\lambda,\mu}$.

Definition 3.1. A(λ, μ)-tableau t is said to be standard if the entries of t are all positive integers, and t_λ and t_μ are both standard tableaux. A tabloid {t} is a standard tabloid if there is a standard tableau in the equivalence class {t}. A polytabloid e_t is a standard polytabloid if t is a standard tableau.

Theorem 3.2. The set $\{e_t \mid t$ is a standard (λ, μ)-tableau$\}$ is a K-basis for $S^{\lambda,\mu}$. The dimension of $S^{\lambda,\mu}$ is the number of standard (λ, μ)-tableaux.

We will spend this section proving this theorem. In the first place we will establish that the e_t are linearly independent. For this case, we shall need a partial order on tabloids associated with the same pair of partitions of n.

Let $t = (t_\lambda, t_\mu)$ be a (λ, μ)-tableau. Let $|t_\mu|$ denote the modulus of t_μ, that is, if $t_\mu = (t_\mu(i, j))$, where $t_\mu(i, j)$ stands for the entry of t_μ in position (i, j) then $|t_\mu| = (|t_\mu(i, j)|)$.

For example, if

$$t_\mu = \begin{matrix} -1 & 2 & 5 \\ 4 & -6 & \\ -3 & & \end{matrix} \qquad \text{then } |t_\mu| = \begin{matrix} 1 & 2 & 5 \\ 4 & 6 & \\ 3 & & \end{matrix}$$

Given any tableau $t = (t_\lambda, t_\mu)$, let $m_{ir}(t_\lambda)$ denote the number of entries less than or equal to $i (i = \pm 1, ..., \pm n)$ in the first r rows of t_λ, and let $n_{ir}(t_\lambda)$ denote the number of entries less than or equal to $i (i = 1, ..., n)$ in the first r rows of $|t_\mu|$.

Definition 3.3. Let (λ, μ) be a pair of partitions of n. Let {t} and {s} be (λ, μ)-tabloids. Then we write

(a) $\{t_\lambda\} \trianglelefteq \{s_\lambda\}$ if one of the following conditions is satisfied:

1. some entries of t_λ are negative and the entries in s_λ are all positive (in this case $\{t_\lambda\} \lhd \{s_\lambda\}$),

2. for all i $(i = \pm 1, 2, ..., \pm n)$ and r, $m_{ir}(t_\lambda) \leq m_{ir}(s_\lambda)$, and

(b) $\{t_\mu\} \trianglelefteq \{s_\mu\}$ if for all $i (1 \leq i \leq n)$ and r, $n_{ir}(t_\mu) \leq n_{ir}(s_\mu)$.

Thus, {s} dominates {t}, written $\{t\} \trianglelefteq \{s\}$, if $\{t_\lambda\} \trianglelefteq \{s_\lambda\}$ and $\{t_\mu\} \trianglelefteq \{s_\mu\}$.

Let t be a (λ, μ)-tableau. Suppose that a is in the p-th row and b is the q-th row of t_γ, where $a = \pm k$, $b = \pm \ell$ $(1 \leq k, \ell \leq n)$ and $\gamma = \lambda$ or μ. If $\gamma = \lambda$ and $a < b$ then the definition of $m_{ir}(t_\lambda)$ gives

$$m_{ir}\big((a, b)(-a, -b)t_\lambda\big) - m_{ir}\big(t_\lambda\big) = \begin{cases} 1 & \text{if } q \leq r < p \text{ and } a \leq i < b \\ -1 & \text{if } p \leq r < q \text{ and } a \leq i < b \\ 0 & \text{otherwise} \end{cases}$$

If $\gamma = \mu$ and $k < \ell$ then the definition of $n_{ir}(t_\mu)$ gives

$$n_{ir}\big((a, b)(-a, -b)t_\mu\big) - n_{ir}\big(t_\mu\big) = \begin{cases} 1 & \text{if } q \leq r < p \text{ and } k \leq i < \ell \\ -1 & \text{if } p \leq r < q \text{ and } k \leq i < \ell \\ 0 & \text{otherwise} \end{cases}$$

Therefore we have the following lemma.

Lemma 3.4. (Dominance Lemma for Tabloids) Let {t} be a (λ, μ)-tabloid. Suppose there exist a, b in t_γ such that a appears in a lower row than b, where $a = \pm k$, $b = \pm \ell$ $(1 \leq k, \ell \leq n)$. Then $\{t\} \triangleleft (a, b)(-a, -b)\{t\}$ if one of the following conditions is satisfied:

(i) $\gamma = \lambda$ and $a < b$,
(ii) $\gamma = \mu$ and $k < l$.

The previous lemma puts a restriction on which tabloids can appear in a standard polytabloid, that is

Corollary 3.5. If t is standard and {s} appears in e_t, then $\{s\} \trianglelefteq \{t\}$.

Proof. Let $s = \pi t$, where $\pi \in C_t$, so that {s} appears in e_t. We induct on the number of column inversions in s. We know that a column permutation of t permutes the elements in each column of t and may change the sign of elements in t_λ. Thus we consider all possible cases in terms of $\pi \in C_t$:

Let $\pi = \sigma\tau$ be such that $\tau \in S_n$ and $s = \prod_i w_i \in C^n_2$, where each i appears in t_λ.

(i) Firstly, if $\tau = e$ then $\pi = \sigma$. Thus, since $s = \sigma t$, σ changes the sign of some elements in t_λ and the entries in t_μ do not change. Then $\{s_\lambda\} \lhd \{t_\lambda\}$ by part (a) (1) of Definition 3.3 and $\{s_\mu\} = \{t_\mu\}$. Therefore, $\{s\} \lhd \{t\}$.

(ii) Secondly, if $\sigma = e$ then $\pi = \tau$. Thus, since $s = \tau t$, then in the same column of s_γ there are positive integers $a < b$ such that a is in a lower row than b, where $\gamma = \lambda$ or μ. Thus, by Lemma 3.4 $\{s\} \lhd (a, b)(-a, -b)\{s\}$. Since $(a, b)(-a, -b)\{s\}$ is involved in e_t and $(a, b)(-a, -b)\{s\}$ has fewer inversions than $\{s\}$, induction shows that $(a, b)(-a, -b)\{s\} \lhd \{t\}$. Therefore, $\{s\} \lhd \{t\}$.

(iii) Finally, if $\tau, \sigma \neq e$ then $\pi = \sigma\tau$ and $s = \pi t$. By (i), some entries of s_λ are negative. Then we have at once $\{s_\lambda\} \lhd \{t_\lambda\}$ by part (a) (1) of Definition 3.3. On the other hand, $\{s_\mu\} \unlhd \{t_\mu\}$ by (ii). Therefore $\{s\} \lhd \{t\}$. Hence, we have the required result.

The previous corollary says that $\{t\}$ is the maximum tabloid in e_t.

Lemma 3.6. Let $\upsilon_1, \ldots, \upsilon_k$ be elements of $M^{\lambda,\mu}$. Suppose, for each υ_i, we can choose a tabloid $\{t_i\}$ appearing in υ_i such that

1. $\{t_i\}$ is maximum in υ_i, and
2. the $\{t_i\}$ are all distinct.

Then $\upsilon_1, \ldots, \upsilon_k$ are linearly independent over K.

Proof. Choose the labels such that $\{t_1\}$ is maximal among the $\{t_i\}$. Thus conditions 1 and 2 ensure that $\{t_1\}$ appears only in υ_i. (If $\{t_1\}$ occurs in υ_i, $i > 1$, then $\{t_1\} \lhd \{t_i\}$, contradicting the choice of $\{t_1\}$.) It follows that in any linear combination

$$c_1\upsilon_1 + c_2\upsilon_2 + \ldots + c_k\upsilon_k = 0$$

we must have $c_1 = 0$ since there is no other way to cancel $\{t_1\}$. By induction on k, the rest of the coefficients must also be zero.

Now we have all the ingredients to prove independence of the standard basis.

Proposition 3.7. The set $\{e_t \mid t$ is a standard (λ, μ)-tableu$\}$ is linearly independent over K.

Proof. By Corollary 3.5, $\{t\}$ is maximum in e_t, and by hypothesis they are all distinct. Thus Lemma 3.6 applies.

We shall next show that standard (λ, μ)-polytaboids span $S^{\lambda,\mu}$. In order to do this, we determine the Garnir element of t associated with e_t by using the following relations.

Lemma 3.8. Let t be a (λ, μ)-tableau.

(i) For any two entries a and b in the same column of t, (a, b) (-a, -b) generates a subgroup of order 2 of C_t. Hence

$(e + (a, b)\ (-a, -b))e_t = 0$ (alternacy relation)

(ii) For any entry a in the tableau t

$(e + \beta(a, -a))e_t = 0$ (sign change relation)

where $\beta = 1$ if a appears in t_λ and $\beta = -1$ if a appears in t_μ.

Proof. The proof following immediately from (1.2) (i).

Remark 3.9. The previous lemma says that we can find the elements of O_n which remove any negative entries of t associated with e_t (sign change relation) and which make t column standard (alternacy relation), i.e. the tableau t associated with e_t may be reorganized so that no negative entries remain and all columns are standard.

Example 3.10. If we take

$$t = \left(\begin{matrix} 1 & 3 \\ 4 & -2 \end{matrix} , \begin{matrix} 5 & 6 \\ 7 & \end{matrix} \right)$$

then

$$e_t = -(2, -2)e_t = e\left(\begin{matrix} 1 & 3 \\ 4 & 2 \end{matrix} , \begin{matrix} 5 & 6 \\ 7 & \end{matrix} \right) \quad \text{(sign change relation)}$$

$$=(2,3)\ (-2, -3)\ e\left(\begin{matrix} 1 & 3 \\ 4 & 2 \end{matrix} , \begin{matrix} 5 & 6 \\ 7 & \end{matrix} \right)$$

$$= e\left(\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix} , \begin{matrix} 5 & 6 \\ 7 & \end{matrix} \right) \quad \text{alternacy relation}$$

Now we want to find elements of the group algebra of O_n which annihilate the given polytabloid e_t. Let t be a (λ, μ)-tableau associated with e_t such that the entries of t were reorganized by the sign change and alternacy relation. Suppose that t has entries t(i, j, k) and t(i,j + 1, k), where k = 1 or 2. Let A be a set of entries in column j of t below and including t(i, j, k), and let B be the set of entries in column j + 1 of t above and including t(i, j + 1, k). Let $A = \{a_i : i \in I\}$ and $B = \{b_j : j \in J\}$, where I and J are index sets. Let S_A, S_B, $S_{A\cup B}$ be the subgroups of O_n. Let $\sigma_1, \ldots, \sigma_r$ be coset representatives for $S_A \times S_B$ in $S_{A\cup B}$, and let

$$S_{A\cup B} = \biguplus_{j=1}^{r} \sigma_j(S_A \times S_B) \text{ and } G_{A,B} = \sum_{j=1}^{r} (\operatorname{sgn} \sigma_j)\, \sigma_j .$$

The element $G_{A,B}$ is called a Garnir element of t associated with e_t.

Remark 3.11. (i) If the elements of A and B belong to t_λ, then we assume that S_A, S_B, $S_{A\cup B}$ are the subgroups of O_n generated by the elements in A, B, $A\cup B$ respectively. Let $C_{A\cup B}$ $(\subset C^2{}_n)$ be the subgroup of $S_{A\cup B}$ generated by $\{w_{a_i}, w_{b_j} \mid i \in I, j \in J\}$. Since $C_{A\cup B}$ is also a subgroup of $S_A \times S_B$, all the coset representatives $\sigma_1, \ldots, \sigma_r$ are positive permutations.

(ii) If the elements of A and B belong to the t_μ, then we assume that S_A, S_B, $S_{A\cup B}$ are the subgroups of O_n generated by positive permutations on A, B, $A\cup B$ respectively.

(iii) The coset representatives $\sigma_1, \ldots, \sigma_r$ are, of course, not unique, but for practical purposes note that we may take $\sigma_1, \ldots, \sigma_r$ so that $\sigma_1 t, \ldots, \sigma_r t$ are all the tableaux which agree with t except in the positions occupied by $A\cup B$, and whose entries increase vertically downwards in the positions occupied by $A\cup B$.

Example 3.12. Let t be as in Example 3.10. Then

$$e_t = e\begin{pmatrix} 1\ 2 & & 5\ 6 \\ & , & \\ 4\ 3 & & 7 \end{pmatrix}$$

Let A = {4} and B = {2, 3}, and let e, (34), (234) be coset representatives for $S_A \times S_B$ in $S_{A\cup B}$. Then

$$G_{A,B} = e - (34) + (234).$$

Let H be any subset of O_n. Define

$$\overline{H} = \sum_{\sigma \in H} (\text{sgn } \sigma)\sigma$$

and if $H = \{\sigma\}$ then we write $\overline{\sigma} = (\text{sgn } \sigma)\sigma$ for $\overline{H}$.

Lemma 3.13. Let $H \leq O_n$ be a subgroup.

(i) If the positive transposition (a, b) (-a, -b) $\in$ H then we can factor $\overline{H} = k(e-(a, b)(-a, -b))$, where $k \in KO_n$.

(ii) If t is a (λ, μ)-tableau with a, b in the same row of t_γ, where $\gamma = \lambda$ or μ, and (a, b) (-a, -b) $\in$ H, then $\overline{H}\{t\} = 0$.

Proof. (i) Consider the subgroup K = {e, (a, b) (-a, -b)} of H. select coset representatives $\sigma_1, ..., \sigma_s$ for K in H and write $H = \biguplus_{i=1}^{s} \sigma_i K$. But then.

$$\overline{H} = \left(\sum_{i=1}^{s} \overline{\sigma_i}\right) (e - (a, b)(-a, -b)) ,$$

as desired.

(ii) By hypothesis, (a, b) (-a, -b) {t} = {t}. Thus,

$$\overline{H}\{t\} = k(e - (a, b)(-a, -b))\{t\} = k(\{t\} - \{t\}) = 0.$$

Proposition 3.14. Let t, A, B be as in the definition of a Garnir element. If $|A \cup B|$ is greater than the number of elements in column j of t then $G_{A,B} e_t = 0$.

Proof. Let

$$\overline{S_A \times S_B} = \sum_{\sigma \in S_A \times S_B} (\text{sgn } \sigma)\sigma \quad \text{and} \quad \overline{S_{A \cup B}} = \sum_{\sigma \in S_{A \cup B}} (\text{sgn } \sigma)\sigma .$$

Consider any $\sigma \in C_t$. If the elements of A and B are in the t_λ then by the hypothesis there must be positive integers a, b $\in A \cup B$ such that c, d belong to the same row of σt_λ where $c = \pm a$, $d = \pm b$. Thus by Remark 3.11 (i) we have (c, d) (-c, -d) $\in S_{A \cup B}$. If the elements of A and B are in the t_μ, then by the hypothesis there must be positive integers a, b $\in A \cup B$ such that a, b belong to the same row of σt_μ. Therefore, (a, b) (-a, -b) $\in S_{A \cup B}$ by Remark 3.11 (ii).

In terms of the element of A and B appearing in the t_γ, where $\gamma = \lambda$ or μ, (c, d) (-c, -d) or (a, b) (-a, -b) belong to the same group $S_{A\cup B}$. But then $\overline{S_{A\cup B}}\ \{\sigma t\} = 0$ by Lemma 3.13. Since this is true for every σ appearing in κ_t, we have $\overline{S_{A\cup B}}\ e_t = 0$.

Now $\overline{S_{A\cup B}} = \biguplus_{j=1}^{r} \sigma_j(S_A \times S_B)$ so $\overline{S_{A\cup B}} = G_{A,B}\overline{S_A \times S_B}$ by Remark 3.11. Since $S_A \times S_B \subset C_t$ then $\overline{S_A \times S_B}$ is a factor of κ_t and $\overline{S_A \times S_B}e_t = |S_A \times S_B|e_t$. Therefore,

$$0 = \overline{S_A \times S_B}e_t = |S_A \times S_B|G_{A,B}e_t$$

Thus, $G_{A,B}e_t = 0$ when the base field is **Q**, and since all the tabloid coefficients here are integers the same holds over any field K.

Example 3.15. Referring to Example 3.10 and Example 3.12, we have

$$0 = G_{A,B}e_{t'} = e_{t'} - e_{(34)t'} + e_{(234)t'} ,$$

where $t' = \begin{pmatrix} 1 & 2 & & 5 & 6 \\ & & , & & \\ 4 & 3 & & 7 & \end{pmatrix}$, so

$$e_t = e_{t'} = e_{(34)t'} - e_{(234)t'} .$$

We shall now use the Garnir relations, sign change relation and alternacy relation to prove that any polytabloid can be written as a linear combination of standard polytabloids. We have already shown how to do this in Example 3.15.

Theorem 3.16. The set $\{e_t \mid t$ is a standard (λ, μ)-tableau$\}$ spans $S^{\lambda,\mu}$.

Proof. First we write [t] for the column equivalence class of t, that is

$$[t] = \{t' \mid t' = \sigma t \text{ for some } \sigma \in C_t\}.$$

The column equivalence classes are partially ordered in a way similar to the partial order in Definition 3.3 on the row equivalence classes.

Let t be any (λ, μ)-tableau associated with e_t. Then we may assume that e_t may be written as a linear combination of column standard polytabloids by Lemma 3.8. Therefore, because of Remark 3.9, we may always take t to have increasing columns and no negative entries.

Suppose that t is not standard. This means that t_γ is not standard, where $\gamma = \lambda$ or μ. By introduction, we may assume that e_s can be written as a linear combination of the polytabloids $e_{t'}$ such that each t'_γ is standard, where $\gamma = \lambda$ or μ, when $[s] \trianglerighteq [\tau]$ and prove the same result for e_t by considering t_γ, where $\gamma = \lambda$ or μ.

Now suppose that $\gamma = \lambda$. Then there must be some adjacent pair of columns in the part t_λ of t, say the j-th and (j + 1)-th columns, with entries $a_1 < a_2 < \dots < a_p$ and $b_1 < b_2 < \dots < b_q$ with $a_i > b_i$ for some i. Thus we have the following situation in the part t_λ of t:

$$\begin{array}{ccc}
a_1 & & b_1 \\
 & & \wedge \\
a_2 & & b_2 \\
 & & \wedge \\
\vdots & & \vdots \\
 & & \wedge \\
a_i & > & b_i \\
\wedge & & \cdot \\
a_{i+1} & & \cdot \\
 & & \cdot \\
\wedge & & b_q \\
\vdots & & \\
\wedge & & \\
a_p & &
\end{array}$$

Take $A = \{a_i, a_{i+1}, \dots, a_p\}$ and $B = \{b_1, b_2, \dots, b_i\}$, and consider the corresponding Garnir element $G_{A,B} = \sum_\sigma (\mathrm{sgn}\ \sigma)\sigma$. By Proposition 3.14, we have $G_{A,B}e_t = 0$ so that

$$e_t = - \sum_{\sigma \neq e} (\mathrm{sgn}\ \sigma) e_{\sigma t}$$

But $b_1 < b_2 < \ldots < b_i < a_i < \ldots < a_p$ implies that $[\sigma t] \trianglerighteq [t]$ for $\sigma \neq e$ by the column analogue of the dominance lemma for tabloids (Lemma 3.4). Since $e_t = - \sum_{\sigma \neq e} (\mathrm{sgn}\ \sigma) e_{\sigma t}$, the result follows from our induction hypothesis for the part t_λ of t.

If the part t_μ of t is also not standard then the result can be deduced when we repeat the above process by considering the part t_μ of t.

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