Commun. Fac. Sci. Univ. Ank. Series A1
V. 47. pp. 85-96 (1998)

SOME BOUNDS FOR THE VARIANCE FUNCTION IN RENEWAL PROCESSES

HALİL AYDOĞDU and FİKRİ ÖZTÜRK

Ankara University, Faculty of Sciences, Department of Statistics, Tandoğan, Ankara, TURKEY

(Received Sep. 15, 1997; Accepted May 25, 1998)

ABSTRACT

In general it is impossible to obtain analytical expressions for the renewal function M(t) and the variance function V(t) in renewal processes. Existence of bounds for M(t) and V(t) is of great value. In this study some linear bounds for the variance function are considered.

1. INTRODUCTION

Let $(X_n)_{n=1,2,...}$ be a sequence of independent and identically distributed nonnegative random variables representing the successive lifetimes with distribution function F. Assume that F has positive mean μ ($F(0) < 1$) and finite variance σ^2. For $S_0 = 0$ and $S_n = X_1+...+X_n$, $n=1,2,...$

$$N(t) = \sup\{n: S_n \leq t\} \ , \ t \geq 0$$

is the number of renewals up to time t.

The mean value function (renewal function) of the renewal process $\{N(t), t \geq 0\}$ is

$$M(t) = E(N(t)) = \sum_{n=1}^{\infty} P(S_n \leq t) = \sum_{n=1}^{\infty} F^{n^*}(t) \quad , \quad t \geq 0, \tag{1}$$

where F^{n^*} is the n-fold Stieltjes convolution of F.

It is well known that the renewal function M(t) satisfies the integral equation (renewal equation)

$$M(t) = F(t) + \int_0^t M(t-x)\, dF(x) \quad , \quad t \geq 0 \ . \tag{2}$$

For the renewal function M(t), an asymptotic expression is

$$\lim_{t\to\infty} \left(M(t) - \frac{t}{\mu}\right) = \frac{\sigma^2 - \mu^2}{2\mu^2} \tag{3}$$

where F is assumed to be nonarithmetic [6].

Consider the renewal equation (2) for the renewal function M(t). Let the distribution function F be absolutely continuous. Starting with any arbitrary bounded function $M_1(t)$, let,

$$M_{k+1}(t) = F(t) + \int_0^t M_k(t-x)\, dF(x) \quad , \quad k = 1,2,\ldots . \tag{4}$$

Xie [7] has proved that for all t such that $F(t) < 1$, $M_k(t)$ converges pointwise to M(t) as $k \to \infty$. Further, if $M_1(t) \le M_2(t)$ for all $t \le T$ then the convergence of $M_k(t)$ as $k \to \infty$ is monotone, that is, for all k and $t \le T$, $M_k(t)$ satisfies

$$M_1(t) \le \ldots \le M_k(t) \le M_{k+1}(t) \le \ldots \le M(t). \tag{5}$$

If $M_1(t) \ge M_2(t)$ for all $t \le T$ then the inequalities in (5) are reversed [7].

The variance function of the renewal process $\{N(t),\ t \ge 0\}$ is

$$V(t) = E(N^2(t)) - M^2(t) = M(t)(1 - M(t)) + 2M{*}M(t) \ , \ t \ge 0, \tag{6}$$

where $M{*}M(t) = \int_0^t M(t-x)\, dM(x)$.

It can be shown that the variance function V(t) satisfies the integral equation

$$V(t) = M(t) + M{*}F(t) - M^2(t) + M^2{*}F(t) + \int_0^t V(t-x)\, dF(x) \quad , \quad t \ge 0,$$

see Aydoğdu [1].

Two asymptotic results for the variance function V(t) are as follows [6].

$$\lim_{t\to\infty} \frac{V(t)}{t} = \frac{\sigma^2}{\mu^3} \tag{7}$$

provided $\sigma^2 < \infty$, and

$$\lim_{t\to\infty} \left(V(t) - \frac{\sigma^2}{\mu^3} t\right) = \frac{5\sigma^4}{4\mu^4} - \frac{2\mu_3}{3\mu^3} + \frac{1}{12} \tag{8}$$

provided $\mu_3 = E(X - \mu)^3 < \infty$ and it is assumed that for some $n \geq 1$, F^{n*} possesses a nonnull absolutely continuous component. Of course, in practical applications it is usually dealt with a totally absolutely continuous F and these technical assumptions present no obstacle.

When the distribution function F is the exponential distribution with scale parameter $\lambda > 0$, that is, $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$, the renewal function M(t) and the variance function V(t) corresponding to this distribution are respectively

$$M(t) = \lambda t \ , \ t \geq 0$$

and

$$V(t) = \lambda t \ , \ t \geq 0. \tag{9}$$

In general it is impossible to obtain analytical expressions for the renewal function M(t) and the variance function V(t). In such a case existence of bunds for M(t) and V(t) is of great value. In section 2, some bounds for M(t) obtained by Marshall [5], Barlow and Proschan [2], Brown [4] and Xie [7] will be briefly reminded. In section 3, at first some bounds for V(t) dependent on M(t) and next linear bounds in the form of $at + b$ will be given.

2. SOME BOUNDS FOR THE RENEWAL FUNCTION

From (1) or the renewal equation (2) it is easily seen that

$$F(t) \leq M(t) \leq \frac{F(t)}{1 - F(t)} \quad , \quad t \geq 0. \tag{10}$$

Marshall [5] gives the following linear bounds,

$$\lambda t + b_l \leq M(t) \leq \lambda t + b_u \quad , \quad t \geq 0 \tag{11}$$

where

$$\lambda = \frac{1}{\mu} ,$$

$$b_l = \inf_{t\geq 0} \frac{F(t) - F_e(t)}{\bar{F}(t)} , \tag{12}$$

$$b_u = \sup_{t\geq 0} \frac{F(t) - F_e(t)}{\bar{F}(t)} , \tag{13}$$

$$\bar{F}(t) = 1 - F(t)$$

and $F_e(t) = \frac{1}{\mu} \int_0^t \bar{F}(x)\, dx$ is the equilibrium (or excess) distribution function.

If F has a density f and $r(t) = \frac{f(t)}{\bar{F}(t)}$ being the failure rate function corresponding to F, for $\alpha \leq r(t) \leq \beta$ ($\alpha \geq 0$, $\beta \geq 0$), Barlow and Proschan [2] shows that

$$\lambda t + \lambda/\beta - 1 \leq M(t) \leq \lambda t + \lambda/\alpha - 1.$$

Xie [7] gives some bounds related to F such as, if $b \leq 0$ then for all $t \geq 0$

$$\int_t^\infty \bar{F}(x)\, dx \,/\, \bar{F}(t) \geq \frac{1 + b}{\lambda} \Rightarrow M(t) \geq \lambda t + b$$

and if $b \geq 0$ then for all $t \geq 0$

$$\int_t^\infty \bar{F}(x)\, dx \,/\, \bar{F}(t) < \frac{1 + b}{\lambda} \Rightarrow M(t) \leq \lambda t + b,$$

where $\int_t^\infty \bar{F}(x)\, dx \,/\, \bar{F}(t)$ is the mean residual life at time t.

From the above results it is easily seen that a sufficient condition for M(t) not to cross the asymptote $t/\mu + (\sigma^2 - \mu^2)/2\mu^2$ in (3) is that

$$\int_t^\infty \bar{F}(x)\, dx \,/\, \bar{F}(t) \geq (\leq) \frac{\mu^2 + \sigma^2}{2\mu} \quad , \quad \text{for all } t \geq 0,$$

if $\sigma/\mu \le (\ge)\ 1$, [7].

Finally let us remind some bounds obtained by Brown [4].

If F is an increasing mean residual life (IMRL) distribution function and $\mu_{k+2} = E(X^{k+2}) < \infty$ for an integer $k \ge 0$ then

$$\frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} - \min_{0\le i\le k} c_i t^{-i} \le M(t) \le \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2}$$

where $c_0 = \dfrac{\mu^2}{2\mu^2} - (\bar{F}(0))^{-1}$ and

$$0 \le c_i = -i \int_0^\infty u^{i-1}(M(u) - u/\mu - \mu_2/2\mu^2 + 1)\, du$$

$$= \int_0^\infty u^i d(M(u) - (u-\mu)/\mu) < \infty \ , \quad i = 1,\dots,k.$$

The term c_i is a function of $\mu_1, \dots, \mu_{i+2}$, $i = 1,\dots,k$ which can be recursively computed as

$$c_i = \left[\mu_{i+2}/(i+1)(i+2)\mu^2\right] - \left[\mu_2\mu_{i+1}/2(i+1)\mu^3\right] - \mu^{-1} i! \sum_{s=1}^{i-1} (c_s/s!)\mu_{i+1-s}/(i+1-s)! \ , \ i=1,\dots,k \ .$$

3. SOME BOUNDS FOR THE VARIANCE FUNCTION

From the expression (6) for V(t) it follows that

$$M(t) - M^2(t) \le V(t) \le M(t) + M^2(t) \quad , \quad t \ge 0$$

because $0 \le M{*}M(t) \le M^2(t)$.

We now give some linear bounds in the form of $at + b$ for the variance function V(t).

Theorem 1.

(i) Let $a \ge \frac{1}{\mu}$ and b ($b \ge 0$) be a constant such that $\int_t^\infty [\bar{F}(x)/\bar{F}(t)]dx$

$\leq \frac{1+b}{a}$ for all $t \geq 0$, then

$$V(t) \leq at + b$$

if M(t) is superadditive.

(ii) Let $a \leq \frac{1}{\mu}$ and b ($b \leq 0$) be a constant such that $\int_t^\infty [\bar{F}(x)/\bar{F}(t)]dx \geq \frac{1+b}{a}$ for all $t \geq 0$, then

$$V(t) \geq at + b$$

if M(t) is subadditive.

Proof (i). Let $M_1(t) = at + b$ in (4). Then

$$\begin{aligned} M_2(t) &= F(t) + \int_0^t M_1(t - x)\, dF(x) \\ &= F(t) + \int_0^t (at - ax + b)\, dF(x) \\ &= F(t) + atF(t) + bF(t) - a\int_0^t x\, dF(x) \\ &= F(t) + bF(t) + a\int_0^t F(x)\, dx. \end{aligned}$$

Since $\int_0^t F(x)\, dx = \int_0^\infty \bar{F}(x)\, dx + t - \mu$, $\int_t^\infty \bar{F}(x)\, dx \leq \frac{1+b}{a}\bar{F}(t)$ and $a\mu \geq 1$ we can obtain

$$\begin{aligned} M_2(t) &= F(t) + bF(t) + a\int_t^\infty \bar{F}(x)\, dx + at - a\mu \\ &\leq F(t) + bF(t) + (1+b)\bar{F}(t) + at - 1 \\ &= at + b \\ &= M_1(t). \end{aligned}$$

Hence, it follows from (5) that $M(t) \leq at + b$. So, considering the following result

$$M(t) \text{ is superadditive (subadditive)} \Rightarrow V(t) \leq (\geq)\ M(t) \tag{14}$$

by Barlow and Proschan [3], the proof of (i) is completed. The proof of (ii) can be easily obtained by reversing the inequalities.

The variance function V(t) does not cross the asymptotic lines in (7) and (8). We express these results respectively in the following corollaries.

Corollary 1.1. If $\sigma / \mu \geq (\leq)\ 1$ such that $\int_t^\infty [\bar{F}(x) / \bar{F}(t)]\, dx \leq (\geq)\ \frac{\mu^3}{\sigma^2}$ and M(t) is superadditive (subadditive) for all $t \geq 0$ then $V(t) \leq (\geq) \frac{\sigma^2}{\mu^3} t$.

Corollary 1.2. If $\sigma / \mu \geq (\leq)\ 1$ and $\frac{2\mu_3}{3\mu^3} - \frac{5\sigma^4}{4\mu^4} \leq (\geq)\ \frac{1}{12}$ such that $\int_t^\infty [\bar{F}(x) / \bar{F}(t)]\, dx \leq (\geq)\ \frac{13\mu^3 - 8\mu_3}{12\sigma^2} + \frac{5\sigma^2}{4\mu}$ and M(t) is superadditive (subadditive) for all $t \geq 0$ then $V(t) \leq (\geq) \frac{\sigma^2}{\mu^3} t + \frac{5\sigma^4}{4\mu^4} - \frac{2\mu_3}{3\mu^3} + \frac{1}{12}$.

It is known that if F is a new better than used (NBU) distribution function then M(t) is superadditive and if F is a new worse than used (NWU) distribution function then M(t) is subadditive (Barlow and Proschan [3]). Then, by choosing $a = 1/\mu$ and $b = 0$ in Theorem 1 we obtain the following result.

Corollary 1.3. F is NBU (NWU) $\Rightarrow V(t) \leq (\geq) \frac{t}{\mu}$.

Consider a renewal process $\{N(t),\ t \geq 0\}$ whose interrenewal times have the exponential distribution. Since the exponential distribution function is both NBU and NWU, from Corollary 1.3 we have again the well known result (9), $V(t) = t/\mu$, $t \geq 0$.

Furthermore we easily obtain the following expressions from Corollary 1.3. For $\frac{\sigma}{\mu} \geq (\leq)\ 1$,

$$F \text{ is NBU (NWU)} \Rightarrow V(t) \leq (\geq) \frac{\sigma^2}{\mu^3} t$$

and for $\frac{\sigma}{\mu} \geq (\leq)\ 1$, and $\frac{2\mu_3}{3\mu^3} - \frac{5\sigma^4}{4\mu^4} \leq (\geq) \frac{1}{12}$,

$$\text{F is NBU (NWU)} \Rightarrow V(t) \leq (\geq) \frac{\sigma^2}{\mu^3} t + \frac{5\sigma^4}{4\mu^4} - \frac{2\mu_3}{3\mu^3} + \frac{1}{12}.$$

Using Theorem 3.17 of Barlow and Proschan [3, p. 173] it can be obtained that

$$\text{F is NBUE(NWUE)} \Rightarrow E(N^k(t)) \leq (\geq) \sum_{s=0}^{\infty} \frac{s^k (t/\mu)^s}{s!} e^{-t/\mu}, \quad k = 1,2,..., \tag{15}$$

see Barlow and Proschan [3] for NBUE (NWUE) distributions. Choosing $k = 2$ in (15) we have $E(N^2(t)) \leq (\geq) \frac{t}{\mu} + \frac{t^2}{\mu^2}$. Then it follows immediately from the definition of V(t) that

$$\text{F is NBUE (NWUE)} \Rightarrow V(t) \leq (\geq) \frac{t}{\mu} + \frac{t^2}{\mu^2} - M^2(t). \tag{16}$$

From (11) and (16) we see that

$$\text{F is NBUE} \Rightarrow V(t) \leq \frac{1 - 2b_l}{\mu} t - b_l^2, \quad t \geq -b_l\mu \tag{17}$$

and

$$\text{F is NWUE} \Rightarrow V(t) \geq \frac{1 - 2b_u}{\mu} t - b_u^2, \quad t \geq 0. \tag{18}$$

Theorem 2.

(i) If $b_l \geq 0$ then

$$\frac{1+4b_l-2b_u}{\mu} t + b_l + 2b_l^2 - b_u^2 \leq V(t) \leq \frac{1-2b_l+4b_u}{\mu} t + b_u + 2b_u^2 - b_l^2, \quad t \geq 0.$$

(ii) If $b_l \leq 0$ then

$$V(t) \geq \frac{1+4b_l-2b_u}{\mu} t + b_l + 2b_l b_u - b_u^2, \quad t \geq 0.$$

(iii) If $b_l \leq 0$ and $b_u \geq 0$ then

$$V(t) \leq \frac{1-2b_l+4b_u}{\mu} t + b_u + 2b_u^2 - b_l^2, \quad t \geq -b_l\mu.$$

(iv) If $b_l \le 0$ and $b_u \le 0$ then

$$V(t) \le \frac{1-2b_l+4b_u}{\mu} t + b_u + 2b_l b_u - b_l^2 \quad , \; t \ge -b_l\mu.$$

Proof (i). Let $b_l \ge 0$. Then $b_u \ge 0$. From (11) $b_l M(t) \ge b_l(\lambda t + b_l)$ and $b_u M(t) \le b_u(\lambda t + b_u)$. Thus,

$$V(t) = M(t) - M^2(t) + 2M*M(t)$$

$$\le \lambda t + b_u - (\lambda t + b_l)^2 + 2\int_0^t [\lambda(t - x) + b_u]\, dM(x)$$

$$= \lambda t + b_u - (\lambda t + b_l)^2 + 2\lambda t M(t) + 2b_u M(t) - 2\lambda t M(t) + 2\lambda \int_0^t M(x)\, dx$$

$$\le \lambda t + b_u - (\lambda t + b_l)^2 + 2b_u(\lambda t + b_u) + 2\lambda \int_0^t (\lambda x + b_u)\, dx$$

$$= \frac{1 - 2b_l + 4b_u}{\mu} t + b_u + 2b_u^2 - b_l^2$$

and similarly

$$V(t) \ge \frac{1+4b_l-2b_u}{\mu} t + b_l + 2b_l^2 - b_u^2 .$$

(ii). Let $b_l \le 0$. Then $b_l M(t) \ge b_l(\lambda t + b_u)$. Thus,

$$V(t) = M(t) - M^2(t) + 2M*M(t)$$

$$\ge \lambda t + b_l - (\lambda t + b_u)^2 + 2b_l M(t) + 2\lambda \int_0^t M(x)\, dx$$

$$\ge \lambda t + b_l - (\lambda t + b_u)^2 + 2b_l(\lambda t + b_u) + 2\lambda b_l t + \lambda^2 t^2$$

$$= \frac{1 + 4b_l - 2b_u}{\mu} t + b_l + 2b_l b_u - b_u^2 .$$

(iii). Let $b_l \le 0$ and $b_u \ge 0$. Then $M^2(t) \ge (\lambda t + b_l)^2$, $t \ge -b_l\mu$ and $b_u M(t) \le b_u(\lambda t + b_u)$. Thus,

$$V(t) = M(t) - M^2(t) + 2M*M(t)$$

$$\leq \lambda t + b_u - (\lambda t + b_l)^2 + 2b_u M(t) + 2\lambda \int_0^t M(x)\,dx$$

$$\leq \lambda t + b_u - (\lambda t + b_l)^2 + 2b_u(\lambda t + b_u) + 2\lambda b_u t + \lambda^2 t^2$$

$$= \frac{1 - 2b_l + 4b_u}{\mu} t + b_u + 2b_u^2 - b_l^2 .$$

(iv). When $b_l \leq 0$ and $b_u \leq 0$ the proof is similar to (iii).

As a simple example let us consider the distribution function

$$F(t) = \begin{cases} 0 & , \ t < 0 \\ \frac{5}{7} t & , \ 0 \leq t < 1 \\ 1 - \frac{2}{7} e^{-(t-1)} & , \ t \geq 1 \end{cases}$$

by Marshall [5] for interrenewal times. Here $b_l = -0.14$ and $b_u = 0.08$. Marshall has given the following inequality for the renewal function M(t).

$$1.076t - 0.14 \leq M(t) \leq 1.076t + 0.08 \quad , \quad t \geq 0.$$

We obtain from Theorem 2 (ii) that,

$$V(t) \geq 0.30128t - 0.1688 \quad , \quad t \geq 0$$

and from (iii)

$$V(t) \leq 1.7216t + 0.0732 \quad , \quad t \geq 0.1301.$$

Corollary 2.1.

(i) If F is NBUE then

$$V(t) \geq \frac{1 + 4b_l - 2b_u}{\mu} t + b_l + 2b_l b_u - b_u^2 \ , \ t \geq 0 \tag{19}$$

and

$$V(t) \leq \frac{1 - 2b_l - 4b_u}{\mu} t + b_u + 2b_l b_u - b_l^2 \ , \ t \geq -b_l \mu. \tag{20}$$

(ii) If F is NWUE then

$$\frac{1+4b_l-2b_u}{\mu} t + b_l + 2b_l^2 - b_u^2 \leq V(t) \leq \frac{1-2b_l+4b_u}{\mu} t + b_u + 2b_u^2 - b_l^2 \ , \ t \geq 0. \tag{21}$$

Proof (i). Let F be NBUE. Then from (12) and (13) $b_l \le 0$ and $b_u \le 0$. The proof is completed from Theorem 2 (ii) and (iv).

(ii). Let F be NWUE. Then from (12) and (13) $b_l \ge 0$ and $b_u \ge 0$. The proof is completed from Theorem 2 (i).

When F is NBUE then $\frac{1-2b_l}{\mu} t - b_l^2 \ge \frac{1-2b_l+4b_u}{\mu} t + b_u + 2b_l b_u - b_l^2$ because in this case $b_u \le 0$ and $4b_u \lambda t + b_u + 2b_l b_u \le 0$ for $t \ge -b_l \mu$. When F is NWUE then $\frac{1+4b_l-2b_u}{\mu} t + b_l + 2b_l^2 - b_u^2 \ge \frac{1-2b_u}{\mu} t - b_u^2$ because in this case $b_l \ge 0$ and $4b_l \lambda t + b_l + 2b_l^2 \ge 0$ for $t \ge 0$. Thus the linear bounds given in (20) and (21) are better than (17) and (18).

Let us consider a renewal process with interrenewal times distribution given by the following failure rate function.

$$r(x) = \begin{cases} \frac{1}{2}, & 0 \le x < 1 \\ \frac{1}{3}, & x \ge 1. \end{cases}$$

Clearly F is DFR. At the same time F is IMRL and NWUE. Now $b_l = 0$ and $b_u = r_e(0) / \lim_{t\to\infty} r_e(t) - 1$, where

$$r_e(t) = 1 \Big/ \int_t^\infty \left[\bar{F}(x) / \bar{F}(t)\right] dx = \begin{cases} \frac{1}{2 + e^{(t-1)/2}}, & 0 \le t < 1 \\ \frac{1}{3}, & t \ge 1. \end{cases}$$

$r_e(0) = 0.3837$ and $\lim_{t\to\infty} r_e(t) = \frac{1}{3}$ giving $b_u = 0.1510$, from (11) we can write

$$0.3837t \le M(t) \le 0.3837t + 0.1510 \quad , \quad t \ge 0$$

and from (21)

$$0.2678t - 0.0228 \le V(t) \le 0.6153t + 0.1965 \quad , \quad t \ge 0.$$

REFERENCES

[1] AYDOĞDU, H., "Yenileme Süreçlerinde Tahmin", Unpublished Ph. D. Thesis, Ankara University (1997).

[2] BARLOW, E.R. and PROSCHAN, F., "Comparison of Replacement Policies, and Renewal Theory Implications", Ann. Math. Statist., 35 (1964) 577-589.

[3] BARLOW, E.R. and PROSCHAN, F., Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston, Inc. New York (1975).

[4] BROWN, M., "Bounds, Inequalities and Monotonicity Properties for Some Specialized Renewal Processes", Ann. Prob. 8 (1980) 227-240.

[5] MARSHALL, K.T., "Linear Bounds on the Renewal Function", Siam J. Appl. Math. 24(2) (1973) 245-250.

[6] SMITH, W.L., "Renewal Theory and Its Ramifications", J. Roy. Statist. Soc. B, 20 (1958) 243-302.

[7] XIE, M., "Some Results on the Renewal Equations", Commun. Statist. Theory Meth. 18(3) (1989) 1159-1171.