Commun. Fac. Sci. Univ. Ank. Series A1
V. 46. pp. 93-101 (1997)

PROPERTIES OF 2-DIMENSIONAL SPACE-LIKE RULED SURFACES IN THE MINKOWSKI SPACE $\mathbb{R}_1^n$

İsmail AYDEMİR* - Murat TOSUN** - Nuri KURUOĞLU*

* *Department of Mathematics, Faculty of Educations, Ondokuz Mayıs University, Samsun, TURKEY*

** *Department of Mathematics, Faculty of Arts and Sciences, Sakarya University, Sakarya, TURKEY*

(Received May 26, 1997; Revised Dec. 6, 1997; Accepted Dec. 9, 1997)

ABSTRACT

In this paper we find new characteristic properties for 2-dimensional ruled surface M in $\mathbb{R}_1^n$ and give the sufficient and necessary conditions for which the space-like ruled surface M is to be total geodesic. In addition, some characterisation which is the well-known for the ruled surfaces in the Euclidean 3-space was generalized for the space-like ruled surfaces in $\mathbb{R}_1^n$.

1. INTRODUCTION

We shall assume throughout this paper that all manifolds, maps vector fields, etc... are differentiable of class C^∞. Consider a general submanifold M of the Minkowski space $\mathbb{R}_1^n$. Suppose that, $\overline{D}$ is the Levi-Civita connection of Minkowski space $\mathbb{R}_1^n$, while D is the Levi-Civita connection of Semi Riemann manifold M. If X and Y are the vector fields of M and if V is second fundamental form of M, we have by decomposing $D_X Y$ in a tangential and normal component.

$$\overline{D}_X Y = D_X Y + V(X,Y) \tag{1.1}$$

The equation (1.1) is called Gauss equation, [1].

If ξ is any normal vector filed on M, we find the Weingarten equation by decomposing $\overline{D}_X\xi$ in a tangential and normal component

$$\overline{D}_X\xi = - A_\xi + D^\perp_X\xi. \tag{1.2}$$

A_ξ determines at each point a self-adjoint lineer map and $D^\perp$ is a metric connection in the normal bundle $\chi^\perp(M)$. We use the same notation A_ξ for the linear map and the matrix of the lineer map, [1].

A normal vector field ξ is called parallel in the normal bundle $\chi^\perp$ (M) if we have $D_X^\perp \xi = 0$ for each vector X. If η is a normal unit vector at the point $p \in M$, then

$$G(p,\eta) = \det A_\eta \tag{1.3}$$

is the Lipschitz-Killing curvature of M at p in direction η, [2].

Suppose that X and Y are vector fields on M, while ξ is a normal vector field on $\chi^\perp(M)$. If the standart metric tensor of $\mathbb{R}_1^n$ is doneted by <,> then we have

$$\langle \bar{D}_X Y, \xi \rangle = \langle V(X,Y), \xi \rangle \tag{1.4}$$

and

$$\langle \bar{D}_X Y, \xi \rangle = \langle A_\xi(X), Y \rangle \, . \tag{1.5}$$

From the above equations we obtain

$$\langle V\ (X,Y), \xi \rangle = \langle A_\xi\ (X),\ Y \rangle \tag{1.6}$$

If $\xi_1, \xi_2, \ldots, \xi_{n-2}$ constitute an orthonormal base field of the normal bundle $\chi^\perp(M)$, then we set

$$\langle V\ (X,Y), \xi_j \rangle = V_j\ (X,Y) \tag{1.6}$$

or

$$V(X,Y) = \sum_{j=1}^{n-2} V_j(X,Y)\xi_j. \tag{1.7}$$

The mean curvature vector H of M at the point p is given by

$$H = \sum_{j=1}^{n-2} \frac{trA_{\xi_j}}{2}\xi_j. \tag{1.8}$$

$\|H\|$ is the mean curvature. If H = 0 at each point p of M, then M is said to be minimal, [1].

2. 2-DIMENSIONAL SPACE-LIKE RULED SURFACES IN $\mathbb{R}_1^n$

Let α be a space-like curve and e(s) be a space-like unit vector on the generators in $\mathbb{R}_1^n$. If the space-like base curve α is an orthogonal trajectory of the generators then we get a 2-dimensional ruled surface M. This ruled surface is called 2-dimensional space-like ruled surface and represented by

$$\psi(s,v) = a(s) + v\, e(s).$$

Definition 2.1: Let M be 2-dimensional space-like ruled surface in $\mathbb{R}_1^n$ and V be second fundamental form of M. If V(X,X) = 0 for all X $\in \chi(M)$ then X is called an asymptotic vector field on M.

Theorem 2.1: Let M be 2-dimensional space-like ruled surface in $\mathbb{R}_1^n$. Then the generators of M are asymptotics and geodesics of M.

Proof: Since the generators are the geodesics of $\mathbb{R}_1^n$, we have

$$\overline{D}_e e = 0 .$$

If we set this in the Gauss equation, we get

$$D_e e + V(e,e) = 0 \text{ or } D_e e = -V(e,e).$$

Since $D_e e \in x(M)$ and $V(e,e) \in \chi^{\perp}(M)$ we get $D_e e = 0$ and $V(e,e) = 0$.

Therefore the generators of M are the asymptotics and geodesics of M.

Suppose that $\{e_1,e\}$ is an orthonormal base field of the tangential bundle $\chi(M)$ and $\{\xi_1, \xi_2,..., \xi_{n-2}\}$ is an orthonormal bundle $\chi^{\perp}(M)$. Then we have the following equations.

$$\overline{D}_e \xi_j = a^j_{11} e + a^j_{12} e_1 + \sum_{i=1}^{n-2} b^j_{1i}\xi_i \quad , \quad 1 \le j \le n-2$$

$$\overline{D}_{e_1} \xi_j = a^j_{21} e + a^j_{22} e_1 + \sum_{i=1}^{n} b^j_{2i}\xi_i \quad , \quad 1 \le j \le n-2 \tag{2.1}$$

From these eqations we observe that

$$a^j_{21} = - a^j_{12} \quad , \quad a^j_{11} = 0 \quad , \quad 1 \le j \le n$$

and

$$A_{\xi_j} = -\begin{bmatrix} 0 & a_{12}^j \\ a_{12}^j & a_{22}^j \end{bmatrix}. \tag{2.2}$$

Then we have the following corollary.

Corollary 2.1: The matrix A_{ξ_j} is corresponding to the shape operator of M and A_{ξ_j} is a symmetric matrix in the sense of Lorentz.

Corollary 2.2: The Lipschitz-Killing curvature at $p \in M$ in the direction of ξ_j is given by

$$G(p,\xi_j) = -(a_{12}{}^j)^2.$$

From (2.1) we have

$$a_{12}^j = \langle \bar{D}_e \xi_j, e_1 \rangle = -\langle \xi_j, \bar{D}_e e_1 \rangle \tag{2.3}$$

and

$$\langle \bar{D}_e e_1, e \rangle = -\langle e_1, \bar{D}_e e \rangle = 0 \tag{2.4}$$

while

$$\langle \bar{D}_e e_1, e_1 \rangle = -\langle e_1, \bar{D}_e e_1 \rangle = 0 . \tag{2.5}$$

From (2.4) and (2.5) we observe that

$\bar{D}_e e_1 \in \chi^{\perp}(M)$ or $\bar{D}_e e_1 = V(e,e_1)$. Because of (2.3) we have

$$\bar{D}_e e_1 = V(e,e_1) = \sum_{j=1}^{n-2} \varepsilon_j \langle \xi_j, \bar{D}_e e_1 \rangle \xi_j = -\sum_{j=1}^{n-2} \varepsilon_j a_{12}^j \xi_j \tag{2.6}$$

$$\varepsilon_j = \langle \xi_j , \xi_j \rangle = \begin{cases} -1, & \xi_j \text{ time-like} \\ 1, & \xi_j \text{ space-like} . \end{cases}$$

Because of (1.4) and (2.1) we find

$$a_{22}^j = \langle \bar{D}_{e_1} \xi_j, e_1 \rangle = -\langle A_{\xi_j}(e_1), e_1 \rangle = -\langle V(e_1,e_1), \xi_j \rangle \tag{2.7}$$

and

$$\operatorname{tr} A_{\xi_j} = -a_{22}^j = \langle V(e_1,e_1), \xi_j \rangle \quad , \quad 1 \leq j \leq n-2. \tag{2.8}$$

Theorem 2.2: Let M be 2-dimensional space-like ruled surface in $\mathbb{R}_1^n$ and $\{e_1,e\}$ be the orthonormal base field of the tangential bundle $\chi(M)$. Then the Gauss curvature G can be given as follows

$$G = \langle \bar{D}_e e_1, \bar{D}_e e_1 \rangle.$$

Proof: Let R be the Riemannian curvature tensor field of M. In this case we get

$$G = \langle R(e_1,e)\, e, e_1 \rangle \ , [3] \ . \tag{2.9}$$

By combining (2.9) and $V(e,e) = 0$ we are faced with

$$G = \langle V(e,e_1),\ V(e,e_1) \rangle \tag{2.10}$$

or

$$G = \langle \bar{D}_e e_1, \bar{D}_e e_1 \rangle.$$

From the above Theorem 2.2 Corollary 2.2 and the equation (2.6) we have the following corollaries.

Corollary 2.3: The Gauss curvature of M with respect to the elements of A_{ξ_j}.

$$G = \sum_{j=1}^{n-2} \varepsilon_j \left(a_{12}^j\right)^2 . \tag{2.11}$$

Corollary 2.4: A space-like ruled surface M is developable if and only if the Lipschitz-Killing curvture is zero at each point.

Theorem 2.3: Let M be a 2-dimensional space-like ruled surface in $\mathbb{R}_1^n$. The mean curvature of M is

$$H = \frac{1}{2}\, \varepsilon_j\ V(e_1,e_1) \ .$$

Proof: From (1.8) we know that

$$H = \sum_{j=1}^{n-2} \frac{\operatorname{tr} A_{\xi_j}}{2}\, \xi_j. \tag{2.12}$$

For the matrix A_{ξ_j} given (2.2) we find

$$\operatorname{tr} A_{\xi_j} = -\, a_{22}^j$$

If we substitude (2.8) in (1.8) we get

$H = \frac{1}{2} \varepsilon_j \, V(e_1, e_1)$.

Theorem 2.4: Let M be 2-dimensional space-like ruled surface in $\mathbb{R}_1^n$. M is developable and minimal iff M is total geodesic.

Proof: We assume that M is developable and minimal. If $X,Y \in \chi(M)$, we have $X = ae + be_1$ and $Y = ce + de_1$.

Therefore we get

$$V(X,Y) = ac\ V(e,e) + (ad + bc)\ V(e,e_1) + bd\ V(e_1,e_1).$$

Because of Theorem 2.1 and minimality of M we have $V(e,e) = 0$ and $V(e_1,e_1) = 0$. Moreover, since M is developable $D_e e_1 = 0$. Thus we can write $V(e,e_1) = 0$ and $V(X,Y) = 0$ for all $X,Y \in \chi(M)$.

Now suppose that $V(X,Y) = 0$, $\forall\ X, Y \in \chi(M)$. Then we have $V(e,e) = 0$, $V(e,e_1) = 0$. Because of Theorem 2.1 we have

$$\langle \bar{D}_e e_1, e \rangle = 0 \text{ and } \langle \bar{D}_e e_1, e_1 \rangle = 0 .$$

This means that $\bar{D}_e e_1$, is a normal vector field or $\bar{D}_e e_1 = V(e,e_1)$.

Therefore we have $\bar{D}_e e_1 = 0$. This implies that M is developable and $V(e,e_1) = 0$ implies that M is minimal.

Let M be 2-dimensional space-like ruled surface in $\mathbb{R}_1^n$ and e be unit space-like vector field of the generator. Then we have the following equations of covariant derivative of the orthonormal base field $\{e, e_1, \xi_1, \xi_2, \dots \xi_{n-2}\}$.

$$\bar{D}_{e_1} e_1 = c_{11}e_1 + c_{12}e + c_{13}\ \xi_1 + \dots + c_{1n}\xi_{n-2}$$

$$\bar{D}_{e_1} e = c_{21}e_1 + c_{22}e + c_{23}\ \xi_1 + \dots + c_{2n}\xi_{n-2}$$

$$\bar{D}_{e_1} \xi_1 = c_{31}e_1 + c_{32}e + c_{33}\ \xi_1 + \dots + c_{3n}\xi_{n-2}$$

$$\vdots$$

$$\bar{D}_{e_1} \xi_{n-2} = c_{n1}e_1 + c_{n2}e + c_{n3}\ \xi_1 + \dots + c_{nn}\xi_{n-2}.$$

If we write these equations in the matrix form we get

$$\begin{bmatrix} \overline{D}_{e_1} e_1 \\ \overline{D}_{e_1} e \\ \overline{D}_{e_1} \xi_1 \\ \vdots \\ \overline{D}_{e_1} \xi_{n-2} \end{bmatrix} = \begin{bmatrix} 0 & c_{12} & c_{13} & \cdots & c_{1n} \\ -c_{12} & 0 & c_{23} & \cdots & c_{2n} \\ -\varepsilon_i c_{13} & -\varepsilon_i c_{23} & 0 & \cdots & c_{3n} \\ \vdots & & & & \\ -\varepsilon_i c_{1n} & -\varepsilon_i c_{2n} & -c_{3n} & \cdots & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e \\ \xi_1 \\ \vdots \\ \xi_{n-2} \end{bmatrix} . \tag{2.13}$$

Theorem 2.5: Let M be a 2-dimensional space-like ruled surface in $\mathbb{R}_1^n$. $\{e_1,e\}$ be an orthonormal base field of the tangential bundle $\chi(M)$ and $\alpha(s)$ be an orthonormal trajectory of the generators of M. Then the following propositions are equivalent.

i) M is developable

ii) The Lipschitz-Kiling curvature

$G(p,\xi_j) = 0 \ , \ 1 \leq j \leq n-2$

iii) The Gauss curvature $G = 0$.

iv) In the equation (2.13), $c_{2k} = 0 \ , \ 3 \leq k \leq n$.

v) $A_{\xi_j}(e) = 0$

vi) $\overline{D}_{e_1} e \in \chi(M)$.

Proof: i $\Rightarrow$ ii : We assume that M is developable, since $a^j_{11} = 0$ in (2.1), $1 \leq j \leq n-2$, the Lipschitz-Killing curvature at point p in the direction of ξ_j is given by

$$G(p,\xi_1) = -\left(a^j_{12}(p)\right)^2 = 0 \quad , 1 \leq j \leq n-2 .$$

Because of (2.6) and since M is developable we have

$$\overline{D}_{e_1} e_1 = -\sum_{j=1}^{n-2} \varepsilon_j \left(a^j_{12}\right) \xi_j = 0 .$$

So we find $G(p,\xi_j) = 0 \ , \ 1 \leq j \leq n-2$.

ii $\Rightarrow$ iii : Let $G(p,\xi_j) = 0, \ 1 \leq j \leq n-2$.

Since we have

$$G(p) = - \sum_{j=1}^{n-2} G(p,\xi_j) \quad , \quad \forall\ p \in M$$

we observe that $G = 0$, $\forall\ p \in M$.

iii $\Rightarrow$ iv : Suppose that $G = 0$, $\forall\ p \in M$. Then because of (2.11) we have $a^j_{12} = 0$, $1 \leq j \leq n\text{-}2$. So $\overline{D}_{e_1}\xi_j$ has no component in the direction e. Hence we observe that $c_{2k} = 0$. $3 \leq k \leq n$, in the equation (2.13).

iv $\Rightarrow$ v : Suppose that $c_{2k} = 0$. $3 \leq k \leq n$, in the equation (2.13). That shows that $\overline{D}_{e_1}\xi_j$ has no component in the direction e. Thus we have in the equation (2.1), $a^j_{12} = 0$, $1 \leq j \leq n\text{-}2$.

Moreover, since $a^j_{11} = \langle \overline{D}_{e_1}\xi_j, e\rangle = - \langle \xi_j, \overline{D}_e e\rangle = 0$ and becacuse of the Weingarten equation we find

$$A_{\xi_j}(e) = 0, \ 1 \leq j \leq n\text{-}2.$$

v $\Rightarrow$ vi : Let $A_{\xi_j}(e) = 0$. Then, from the Weingarten equation, we have $a^j_{11} = 0$, $a^j_{12} = 0$, $1 \leq j \leq n\text{-}2$. Moreover, $\langle e, \xi_j\rangle = 0$ implies

$$\langle \overline{D}_{e_1} e, \xi_j\rangle = - \langle e, \overline{D}_{e_1}\xi_j\rangle. \tag{2.14}$$

If we se equations 2.1 and last equations we get

$$\langle \overline{D}_{e_1} e, \xi_j\rangle = - \langle e, \overline{D}_{e_1}\xi_j\rangle = - a^j_{12}$$

and

$$\langle \overline{D}_{e_1} e, \xi_j\rangle = 0 \ .$$

From the last equation we have

$$\overline{D}_{e_1} e \in \chi(M).$$

vi $\Rightarrow$ i : Let $\overline{D}_{e_1} e \in \chi(M)$. Then from the equation (2.14), we get $\langle \overline{D}_{e_1} e, \xi_1\rangle = - a^j_{12} = 0.$ $1 \leq j \leq n\text{-}2$. On the other hand, $e[\langle e_1, e_1\rangle] = e[1]$ implies that $\langle \overline{D}_e e_1, e_1\rangle = 0$ and $e[\langle e_1, e\rangle] = e[0]$ implies that $\langle \overline{D}_e e_1, e\rangle = 0$ (Since the generators are the geodesics of $\mathbb{R}^n_1$, we have $\overline{D}_e e = 0$). Thus $\overline{D}_e e_1 \in \chi(M)$.

Because of (2.6) and since $a^j_{12} = 0$. $1 \leq j \leq n\text{-}2$, we write that $\bar{D}_e e_1 = 0$.

This means tha tangent planes of M constant along the generator e of M. i.e. M is developable.

Corollary 2.5: Let M be a 2-dimensional space-like ruled surface in $\mathbb{R}^n_1$ with a Gauss curvature beign zero. If M is minimal, then $c_{sk} = 0$, $1 \leq s \leq 2$, $3 \leq k \leq n$, in the (2.13).

Proof: Let M be minimal. Then from the eqation (2.12) we have V $(e_1,e_1) = 0$. If this result is set in the Gauss equation, we find

$$\bar{D}_{e_1}e_1 = D_{e_1}e_1 \ .$$

This means that $\bar{D}_{e_1}e_1$ has no component in $\chi^{\perp}(M)$. Therefore we have

$$C_{1k} = 0, \ 3 \leq k \leq n. \ (2.15)$$

in the equation (2.13). On the other hand, since $G = 0$, by hypothesis, and from the Theorem 2.5 we know that $C_{2k} = 0$. $3 \leq k \leq n$. If we consider this together with (2.15) we observe that $C_{sk} = 0$, $1 \leq s \leq 2$, $3 \leq k \leq n$.

REFERENCES

[1] CHEN B.Y., Geometri of Submanifolds, Marcel Dekker, New York 1973.

[2] HOUH, C.S., Surfaces with Maximal Lipschitz-Killing Curvature in the Direction of Mean Curvature Vector, Proc. Amer. Math. Soc. 35(1972) 537-542.

[3] O'NEIL, B., Semi-Riemannian Geometry, Academic Pres, New York, London, 1983.

[4] THAS, C., Een (lokale) Studie van de (m+1)-dimensionale varieteiten, van de n-dimensionale Euclidische Ruimte $\mathbb{R}$ (n ≥ 2m+1 en m ≥ 1), Beschreven door een Eendimensionale Familie van m-dimensionale Lineaire Ruiten. Paleis Der Academien Hertogsstreet, I, Brussel, (1974).

[5] THAS, C., Properties of Ruled Surfaces in the Euclidean Space E^n Academia Sinica Vol 6, No.1, 133-142, 1978.