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SPECIAL CASES OF CAT^1-GROUPS

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ABSTRACT

In this paper we describe a package XMOD [2] of functions for computing with crossed modules, their morphisms and derivations; cat^1-groups, their morphisms and sections, written using the GAP [5] group theory programming language. We have also enumerated the isomorphism classes of cat^1-groups in [3]. Using these papers, we proved some mathematical results on cat^1-group structures.

1. INTRODUCTION

The term crossed module was introduced by J.H.C. Whitehead in [6]. Most references of crossed modules state the axioms of a crossed module using left actions, but we shall use right actions since this is the convention used by most computational group packages. In Section 2 we recall the basic properties of crossed modules and cat^1-groups. In Section 3 we describe he implementation methods of these structures in GAP and in Section 4 we discuss the algorithms used to compute cat^1-groups. Then we proved some mathematical results on cat^1-groups in Section 5.

2. CROSSED MODULES AND CAT^1-GROUPS

In this section we recall the descriptions of two equivalent categories: **X Mod**, the category of crossed modules and their morphisms; **Cat1**, the category of cat^1-groups and their morphisms.

A crossed module $\chi = (\partial : S \rightarrow R)$ consists of a group homomorphism ∂, called the *boundary* of χ, together with an action $\alpha : R \rightarrow \mathrm{Aut}(S)$ satisfying, for all $s, s' \in S$ and $r \in R$,

X Mod 1: $\partial(s^r) = r^{-1}(\partial s)r$
X Mod 2: $s^{\partial s'} = s'^{-1}ss'$.

The standart constructions for crossed modules are as follows:

1. A *conjugation crossed module* in an inclusion of a normal subgroup $S \trianglelefteq R$, where R acts on S by conjugation.

2. A *central extension crossed module* has a boundary a surjection $\partial : S \to R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$.

3. An *automorphism crossed module* has as range a subgroup R of the automorphism group Aut(S) of S which contains the inner automorphism group of S. The boundary maps $s \in S$ to the inner automorphism of S by s.

4. An *R-Module crossed module* has an R-module as source and ∂ is the zero map.

5. The direct product $X_1 \times X_2$ of two crossed modules has source $S_1 \times S_2$, range $R_1 \times R_2$ and boundary $\partial_1 \times \partial_2$, with R_1, R_2 acting trivially on S_2, S_1 respectively.

A morphism between two crossed modules X_1 and X_2 is a pair (σ, ρ), where $\sigma : S_1 \to S_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$\partial_2\sigma = \rho\partial_1, \ \sigma(s^r) = (\sigma s)^{\rho r}.$$

When $X_2 = X_1$ and σ, ρ are automorphisms then (σ, ρ) is an automorphisms of X_1. The group of automorphisms is denoted by $Aut(X_1)$.

In [4] Loday reformulated the notion of a crossed module as a cat^1-group, namely a group G with a pair of homomorphisms $t, h : G \to G$ having a common image R and satisfying certain axioms. We find it convenient to define a cat^1-group $C = (e; t, h : G \to R)$ as a group G, two surjections $t, h : G \to R$ and an embedding $e : R \to G$ satisfying:

Cat 1: $te = he = idR$,
Cat 2: $[\ker t, \ker h] = \{1_G\}$.

The maps t, h are often referred to as the *source* and *target*, but we choose to call them the *tail* and *head* of C, because *source* is the GAP term or the domain of a function. A morphism $C_1 \to C_2$ of cat^1-groups is a pair (γ, ρ) where $\gamma : G_1 \to G_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$h_2\gamma = \rho h_1, \ t_2\gamma = \rho t_1, \ e_2\rho = \gamma e_1. \tag{1}$$

The crossed module X associated to C has $S = \ker t$ and $\partial = h/s$. The cat^1-group associated to X has $G = R \ltimes S$, using the action from X, and

$$t(r, s) = r, \ h(r, s) = r(\partial s), \ er = (r, 1).$$

3. GAP IMPLEMENTATION

The group theory program GAP [5] is designed to facilitate the implementation of new structures as record types with their own output form. A seperate operations record allows the overloading of functions such as identity mapping kernel, etc.. We have developed a package [2] of some 160 functions for crossed modules, their morphisms and derivations; cat^1-groups have permutation groups and finitely presented groups are used in many of the constructions. The underlying groupoid G has not been included, nor does the package [2] contain functions for pre-cat^1-groups and Peiffer subsgroups. The following implementation method was explained very clearly in [3] and [1].

We implement a cat^1-groups $C = (e; t, h : G \to R)$ as a record with fields:

C. source,	the source group G,
C. range,	the range group R,
C. tail,	the tail homomorphism t,
C. head,	the head homomorphism h,
C. embedRange	the embedding e of R in G,

C. kernel,	the permutation group S isomorphic to the kernel of t,
C. embedKernel,	the isomorphism $\varepsilon : S \to \ker t$,
C. boundary,	the restriction ∂ of h to S,
C. isDomain,	set **true**,
C. operations,	a special set of operations **Cat1GroupOps,**
C. name,	a concatenation of the names of G and R,
C. is Cat1 Group,	a boolean flag, normally **true.**

The operations record **Cat1 Ops** includes functions for equality; size and list of elents and a special output form.

A morphisms **mor** = $(\gamma, \rho) : C \to D$ of cat^1-groups is implemented as a record with fields:

mor.source,	the source cat1-group **C**,
mor.range,	the range cat1-group **D**,
mor.sourceHom,	the homomorphism γ from **C.source** to **D.source**,
mor.rangeHom,	the homomorphism ρ from **C.range** to **D.range**,
mor.isCat1Morphism,	a Boolean flag, normally **true**,
mor.operations,	a special set of operations **Cat1MorphismsOps**,
mor.name,	a concatenation of the names of **C** and **D**.

The operations record **Cat1 MorphismOps** includes function for equality; kernel and image; composite and inverse morphism; and tests such as **IsEpimorphism.**

Example 3.1 Let R be the group S_3C_7 and S its normal subgroup S_3. The inclusion crossed module X = $(\iota : S_3 \to S_3C_7)$ and cat^1-groups structure are given as follows.

```
gap > X : ConjugationXMod (s3c7, s3);
Crossed module [s3 -> s3c7]
gap > XModPrint (X);
Crossed module [s3 -> s3c7] : -
: Source group has parent (s3c7) and has generators:
[ (1, 2), (2, 3) ]
: Range group = s3c7 has generators:
[ (1, 2), (2, 3), (4, 5, 6, 7, 8, 9, 10) ]
: Boundary homomorphism maps source generators to:
[ (1, 2), (2, 3) ]
```

```
: Action homomorphism maps range generators to automorphism:
(1, 2) --> source gens --> [ (1, 2), (1, 3) ]
(2, 3) --> source gens --> [ (1, 3), (2, 3) ]
(4, 5, 6, 7, 8, 9, 10) --> source gens --> [ (1, 2), (2, 3) ]
```

These 3 automorphisms generate the group of automorphisms.

```
gap> C : = Cat1XMod (X);
cat1-group [Perm(s3c7 |X s3) ==> s3c7]
gap> Cat1Print (C);
cat1-group [Perm(s3c7 |X s3) ==> s3c7] : -
: source group has generators:
[ (2, 6)(4, 5)(7, 8), (3, 6)(4, 5)(8, 9), (10, 11, 12, 13, 14, 15, 16),
(1, 3)(2, 4)(5, 6), (1, 2)(3, 5), (4, 6) ]
: range group has generators:
[ (1, 2), (2, 3), (4, 5, 6, 7, 8, 9, 10) ]
: tail homomorphism maps source generators to:
[ (1, 2), (2, 3), (4, 5, 6, 7, 8, 9, 10), (1, 2), (2, 3) ]
: range embedding maps range generators to:
[ (2, 6)(4, 5)(7, 8), (3, 6)(4, 5)(8, 9), (10, 11, 12, 13, 14, 15, 16) ]
: kernel has generators:
[ (1, 2), (2, 3) ]
: boundary homomorphism maps generators of kernel to:
[ (1, 2), (2, 3) ]
: kernel embedding maps generators of kernel to:
[ (1, 3)(2, 4)(5, 6), (1, 2)(3, 5)(4, 6) ]
: assoiated crossed module is Crossed module [s3 -> s3c7]
```

4. Algorithms for Cat^1-groups

The following algorithms were explained very clearly in [1].

4.1. Algorithm for Cat1

The function **Cat1** is called as:

```
gap> Cat1 (G, t, h, [e] ).
```

The function requires three parameters and an optional parameter: a group G, and the tail and head homomorphisms t, h and optional embedding homomorphism e. As output, the function returns a cat^1-group with field as described in section 4.

- **Step 1** Check that there are three parameters and that the first argument is a permutation group.
- **Step 2** Check that t and h are homomorphisms with source G and with a common range R.
- **Step 3** Set up the record fields which listed in section 4.
- **Step 4** Call the **IsCat1** function to verify the axioms.

4.2. Algorithm for IsCat1

The function **IsCat1** is called as:

gap> IsCat1 (C)

The function returns **true** when the input parameter C is a cat^1-group and **false** otherwise. The function checks that the main fields of a cat^1-group exist, and that the axioms CAT1 and CAT2 are satisfied.

- **Step 1** Check that C is a record structure, that fields **C.source** and **C.range** exist, and that these are permutation group..
- **Step 2** Check that tail and head exist, and that these are group homomorphisms.
- **Step 3** Check that **C.embedRange**, C.embedKernel, **C.boundary** and **C.kernel** exist.
- **Step 4** Check that the cat^1-group conditions CAT1 and CAT2 are satisfied.
- **Step 5** Add filed **.isCat1** to C.

4.3. Algorithm for Cat1Morphism

The function **Cat1Morphism** is called as:

gap> Cat1Morphism (C, D, homs)

The function **Cat1Morphism** require as parameters two cat^1-groups and a two-element list containing the source and range homomorphisms. As output, it sets up the required fields for morphism μ. The algorithm of this function as same as the algorithm of XModMorphism.

In this implementation a morphism of cat^1-groups is a record with fields as described in section 4.

4.4. Algorithm for Cat1XMod

The function **Cat1XMod** is called as:

gap> Cat1XMod(X)

This function implements the functor **XMod** $\rightarrow$ **Cat1.**

Step 1 Call **IsXMod** on the argument.

Step 2 If **X.action** is trivial then the source group is the constructed using the direct product $G = R \times S$. If **X.action** is not trivial the source group is constructed as a permutation representation G of R x S.

Step 3 The tail, head and embedding are defined using $t(r, s) = r$, $h(r, s) = r\partial s$, $e(r) = (r, 1)$.

Step 4 Create the record structure with the required fields in section 4.

Step 5 Add **X.cat1** and **C.xmod.**

The procedure for **XModCat1** is similar.

5. Special cases for cat^1-structures

By a *cat^1-structure* on G we mean a cat^1-group C where R is a subgroup of G and e is the inclusion map. For such a structure to exist, G must contain a normal subgroup S with $G/S \cong R$. Furhthermore, since t, h are respectively the identity and zero maps on S, since $G \cong R \ltimes S$ we require $R \cap S = \{1_G\}$.

The boundary map ∂ of a cat^1-structure is the zero map if and only if $t = h$ since

$$\ker t = \ker h \Rightarrow \partial = 0 \Rightarrow h'(r,s) = r(\partial s) = r = t'(r,s) \Rightarrow h' = t' \Rightarrow h = t.$$

Furthermore, by Cat2, kert is abelian. This is precisely the situation for any cat^1-group whose source G is a group such that no two normal subgroups are isomorphic. Examples of such groups are cyclic groups, simple groups and symmetric groups.

Proposition 5.1. Let (γ, ρ) be an isomorphism from $C = (e; h, t : G \to R)$ to C' $(e'; h', t' : G' \to R')$. Then $\gamma : G \to G'$, $\rho : R \to R'$ are isomorphisms, $t' = \rho t \gamma^{-1}$, $h' = \rho h \gamma^{-1}$ and $e' = \gamma e \rho^{-1}$.

Proof: This follows from (1) since γ, ρ are invertible.

Proposition 5.2. Up to isomorphism, the only cat^1-structure with $R = G$ is the *identity* cat^1-structure (id; id, id : $G \to G$) on G.

Proof: The condition $R \cap S = \{1_G\}$ implies that $S = \{1_G\}$ and hence $t = h$, $e = t^{-1}$. We then have a cat^1-group isomorphism

$$(t, id) : (t^{-1}; t, t: G \to G) \to (id; id, id : G \to G).$$

Proposition 5.3. The *zero* cat^1-structure $(0; 0, 0 : G \to \{1_G\}$ on G is a cat^1-group if and only if G is abelian.

Proof: When $t = h = 0$ condition **Cat2**: becomes $[\ker t, \ker h] = [G, G] = \{1_G\}$.

Proposition 5.4. Up to isomorphism, the only cat^1-structure on a finite non-abelian simple group is the identity cat^1-group.

Proof: This follows immediately from the previous two Propositions.

Poposition 5.5. Up to isomorphism, the only cat^1-structure on the quaternionic group Q_{2^n} of order 2^n is the identity cat^1-group.

Proof: Let G be a group with normal subgroup N such that the only subgroup of G which does not contain N is $\{1_G\}$. Then the condition $R \cap S = \{1_G\}$ is only satisfied if $R = \{1_G\}$ or $S = \{1_G\}$, so the only possible cat^1-structures are the identity structure and, provided G is abelian, the zero structure.

The quaternion groups Q_{2^n} are non-abelian and of this type, with $N = \{1, -1\}$.

Proposition 5.6. Two *abelian* cat^1-groups C, C′ are isomorphic if and only if $G \cong G'$, $R \cong R'$ and $\ker \partial \cong \ker \partial'$.

Proof: Since G is abelian, we may assume that it has direct sum decomposition of the form $R \oplus S$ where $R = R_0 \oplus R_1$, $S = S_0 \oplus S_1$, $S_0 = \ker \partial$ and $R_0 = \mathrm{im}\partial$. If $G \cong G'$ and $R \cong R'$ then we may decompose $G' = R' \oplus S' = R'_0 \oplus R'_1 \oplus S'_0 \oplus S'_1$. If γ is any isomorphism which maps the four factors of G onto those of G′ then $(\gamma, \gamma_R) : C \to C'$ is an isomorphism.

Proposition 5.7. Let C_n be a cyclic group and let n have exactly k distinct prime factors. Then there are 2^k isomorphism classes of cat^1-structures on C_n.

Proof: Recall that C_n has n endomorphisms $t_s : g \mapsto g^s$, $1 \le s \le n$ where g is a generator of C_n, and that the idempotent endomorphisms are determined as follows. Let $n = p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$ and let $q_i = n/p_i^{m_i}$ for each $1 \le i \le k$. Euclid's algorithm provides an identity $1 = a_1q_1 + \ldots + a_kq_k$ and we set $B = \{b_1, \ldots, b_k\}$ where $b_i = a_iq_i$ reduced modn. Then if $S \subseteq B$ and s is the sum of the elements in S, the endomorphism t_s is idempotent, with image C_{d_s} where $ds = \gcd(n, s)$. The 2^k subsets provide the 2^k idempotent endomorphisms. Since no two subgroups of C_n are isomorphic, it follows from Proposition 5.6 that the isomorphism classes o cat^1-structues on C_n have representatives $(e_s;\ t_s,\ t_s : C_n \to C_{d_s})$.

Proposition 5.8. Let $G = C_p^{\ n}$ be an elemantary abelian group. The number of isomorphism classes of cat^1-structures on G is

$$\begin{cases} (1 + n/2)^2 , & n \text{ even,} \\ (n + 1)(n + 3)/4 , & n \text{ odd.} \end{cases}$$

Proof: When $R \cong C_p^{\ n-m}$ for some $0 \le m \le n$, $\mathrm{im}\partial \in \{I, C_p, C_p^{\ 2}, \ldots, C_p^{\ \min(m,n-m)}\}$, so by Proposition 5.6 there are 1 + min (m,n-m) isomorphism classes for fixed m. Summing over all $m \le n$ gives the required results.

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