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SOME PROPERTIES OF THE CESÀRO OPERATOR IN THE SPACE s_r

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ABSTRACT

We study the spectrum of Cesàro operator in the space s_r, and corresponding matrix equations. Then we study the consequences of the variation of an element in the Cesàro matrix.

1. INTRODUCTION

We are here interested in an application on the infinite matrices theory to the Cesàro operator C, this one being represented by:

$$C = \begin{pmatrix} 1 & & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \mathbf{0} & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & & \\ . & . & . & . & & \\ \frac{1}{n} & \frac{1}{n} & . & . & \frac{1}{n} & \\ . & . & . & . & . & . \end{pmatrix}.$$

Such problem has been studied in Cooke [1] and Hausdorff [3]; recall the well known following result: if $X = (x_n)$ is a sequence which converges to a limit ℓ then the sequence defined by $CX = (y_n)$ image of X by the Cesàro operator converges to the same limit ℓ. Another important property of this operator is that a lower triangular matrix A is permutable with the Cesàro matrix, if and only if $A = H^{-1}DH$, where D is a diagonal matrix and

$$H = \begin{pmatrix} \binom{0}{0} & & \mathbf{0} & \\ \binom{1}{0} & -\binom{1}{1} & & \\ \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} & \\ . & . & . & . \end{pmatrix}.$$

($\binom{n}{k}$ with $0 \le k \le n$, being the number of combinations of n things, k at a time), [3].

The matrices $H^{-1}DH$ are called the Hausdorff matrices.

More recently J.B Reade [13], and J.T. Okutayi [10] gave some results on the spectrum of Cesàro operator, this last being considered as a map from the bounded variation space bv into itself, bv being the space of the convergent sequences $X = (x_n)$ such that $\sum n|x_{n+1} - x_n| < \infty$. These authors proved that this spectrum is the disk $\left|z - \frac{1}{2}\right| \le \frac{1}{2}$. Many other authors have studied infinite matrices in the general case, such as Maddox [6], Malkowsky [8], Mursaleen [9], Petersen [11], [12], Defranza and Zeller [2]

In this paper we study the spectrum of this operator, considered as a map from the Banach space, s_r into itself. s_r have been used in many papers, see R. Labbas B. De Malafosse [4], [5] or B. De Malafosee [7] We study after the properties of the operator $C - \lambda I$, when λ belongs to the spectrum of C. At last, we examine the behavior of the matrix equation, when an element is changed. More particularly, we examine the distance between two solutions, when the index of the row, or of the column, containing the new element tends to infinity.

This paper is organized as follows: In Section 2 we recall the definitions of the spaces s_r and S_r and we give some elementary results concerning the properties of the map C. In Section 3.1 we study the Cesàro spectrum relatively to the space s_r. In Section 3.2 we describe the space of the solutions of equation $(C - \lambda I)X = B$, when λ belongs to the spectrum; this one depending on the second member. Then we describe the image space of $C - \lambda I$, and we give eigenvectors spaces. In Section 4, one studies the behavior of the matrix equation CX=B, when an element of C is changed.

2. FIRST PROPERTIES OF CESÀRO OPERATOR RELATIVELY TO THE SPACES S_r AND s_r.

We study, here, the map $C : X = (x_n) \mapsto CX$, defined naturally by:

$$CX = {}^t\left(x_1, \frac{1}{2}(x_1 + x_2), \cdots, \frac{1}{n}(x_1 + x_2 + \cdots + x_n), \cdots \right)$$

in the space s_r [3], that we recall.

For r > 0, we denote by s_r the Banach space of the one column matrices:

$$s_r = \left\{ X = (x_n) : \sup_n \left(\frac{|x_n|}{r^n} \right) < \infty \right\}, \tag{1}$$

normed by:

$$\|X\|_{s_r} = \sup_n \left(\frac{|x_n|}{r^n} \right) \tag{2}$$

Analogously, we define the Banach algebra S_r, of infinite matrices $A = (a_{nm})$, with unit element $I = (\delta_{nm})$, (that is $\delta_{nm} = 0$ if $n \neq m$, and $\delta_{nm} = 1$) by:

$$S_r = \left\{ A = (a_{nm}) : \sup_n \left(\sum_m |a_{nm}| r^{m-n} \right) < \infty \right\} \tag{3}$$

normed by:

$$\|A\|_{S_r} = \sup_n \left(\sum_m |a_{nm}| r^{m-n} \right)$$

when $A \in S_r$ and $X \in s_r$ then $AX \in s_r$, and:

$$\|AX\|_{s_r} \leq \|A\|_{S_r} \|X\|_{s_r}.$$

One can then, study linear infinite system:

$$\sum_{m=1}^{\infty} a_{nm} x_m = b_n \quad n = 1,2,\ldots; \tag{4}$$

which is equivalent to the equation AX=B, where $A = (a_{nm}), B = (b_n)$ are given infinite matrices, and $X = (x_n)$ is the unknown infinite column matrix. Recall that if

$$\| I - A \|_{S_r} = \sup_n \left(|1 - a_{nn}| + \sum_{m \neq n} |a_{nm}| r^{m-n} \right) < 1,$$

the system (4) admits in s_r a unique solution

$$X = \sum_{n \geq 0} (I - A)^n B. \tag{5}$$

We see, at first, that the Cesàro operator C belongs to any space S_r with R > 1, since:

$$\sup_{n\geq 1}\left[\frac{1}{n}\left(\frac{1}{R^n}+\frac{1}{R^{n-1}}+\ldots+\frac{1}{R}\right)\right]<\infty.$$

Throughout this paper we multiply the matrix A on the left, by a matrix D, (which does not belong necessarily to any space S_r) so that to obtain all the diagonal elements of DA equal to 1, whenever it is possible. This method permits to calculate easily $\|I-DA\|_{S_r}$. For instance, letting $D=(n\delta_{nm})$ simple calculation applied to C permits to obtain: $C^{-1}=(DC)^{-1}D$, that is:

$$C^{-1}=\begin{bmatrix} 1 & & & \\ -1 & 2 & & \mathbf{0} \\ & -2 & 3 & \\ \mathbf{0} & & -n+1 & n \\ & & & \ \ \cdot\ \ \cdot \end{bmatrix},$$

and we see that C^{-1} does not belong to any space S_r. However DC and $(DC)^{-1}$ belong to $S_R, \forall R>1$. That is, for a given sequence $B=(b_n)$ the system:

$$b_n=\frac{1}{n}\left(\sum_{i=1} x_i\right) \qquad (n=1,2,\ldots),$$

where $X=(x_n)$, admits a unique solution, given by: $X=(-(n-1)b_{n-1}+nb_n)_{n\geq 1}$.

We see that if $X=(x_n)\in s_1$ it is the same for $B=(b_n)$ and more generally C maps s_R into itself, for all R > 1, since C belongs to S_R. Note that if $B\in s_1$, we cannot deduct that it is the same for X, since C^{-1} does not map s_1 into itself. In the general case [6], it is written that $A\in(s_1,s_1)$ if A maps s_1 into itself. It has been proved that $A\in(s_1,s_1)$ if and only if $A\in S_1$. One deducts that for any $r>0$, $A\in(s_r,s_r)$ if and only if $A\in S_r$.

3. PROPERTIES OF THE OPERATOR $C-\lambda I$ CONSIDERED AS A MAP FROM s_R INTO ITSELF

3.1 CASE WHERE $\lambda\neq 0$, AND $\lambda\neq\frac{1}{n}$, FOR ALL $n\geq 1$.

As in [9], and [12] where the authors studied the operetor $C-\lambda I$, mapping bv, into itself, we give some properties of this operator relatively to the space $s_R,(R>1)$.

From the preceding, we see that 0 is in the spectrum of C, since the unique solution of equation CX=B, (where B is well chosen in $s_R, R>1$) does not belong to s_R. The matrix $C-\frac{1}{n}I$ is not invertible, since an element of its diagonal is zero, which proves that $\frac{1}{n}$ belongs to the spectrum of C. In the following we shall refer to the matrix C'_λ in which $c_n=\frac{n\lambda}{n\lambda-1}$, $n\geq 2$:

$$C'_\lambda=\begin{pmatrix} 1 & & & & \\ c_2 & 1 & & \mathbf{0} & \\ & \cdot & \cdot & & \\ \mathbf{0} & & c_n & 1 & \\ & & & \cdot & \cdot \end{pmatrix}. \tag{6}$$

We recall [4] the following result, where $J=\{0,1,1/2,...,1/n,...\}$, and λ is a complex number:

Propositon 1. For every $R>1$ and every $\lambda\notin J$, $C-\lambda I$ is bijective from s_R into itself. The spectrum of C is $\sigma_C=J$.

We have seen above, that $J\subset\sigma_C$. To prove that $\sigma_C\subset J$, we need a lemma:

Lemma 2. Let $R>0$ and $\lambda\notin J$, then we have $(C'_\lambda)^{-1}\in S_R$.

Let us prove that $\forall B\in s_R$ $\forall\lambda\notin J$, equation $C'_\lambda X=B$ admits a unique solution in the space s_R. Let $N=\left[\frac{R_{\rho_0}}{|\lambda|(R_{\rho_0}-1)}\right]+1, \rho_0$ being a real, fixed such that $1/R<\rho_0<1$; and consider the finite system:

$$\begin{cases} x_1=b_1, \\ c_n x_{n-1}+x_n=b_n, \quad n=2,\dots,N-1. \end{cases} \tag{7}$$

which admits a unique solution, that we shall denote $x_1^0, x_2^0,\dots,x_{N-1}^0$. Equation $C'_\lambda X=B$ is then equivalent to the infinite system:

$$\begin{cases} x_N=b_N-c_N x_{N-1}^0, \\ c_n x_{n-1}+x_n=b_n, \quad n=N+1,N+2,\dots \end{cases} \tag{8}$$

the first N – 1 coordinates being determined above. (8) can be written under the form $C_\lambda^{'N} X^N = B^N$, where $C_\lambda^{'N}$ is the matrix:

$$\begin{pmatrix} 1 & & & \\ c_{N+1} & 1 & & \mathbf{0} \\ & c_{N+2} & . & \\ \mathbf{0} & & c_{N+n} & 1 \\ & & & . \quad . \end{pmatrix},$$

and

$$^t X^N = (x_N, x_{N+1}, \ldots), \quad ^t B^N = (b_N - c_N x_{N-1}^0, b_{N+1}, \ldots, b_{N+n}, \ldots).$$

The choice of N, permits to assert that:

$$\left\| I - C_\lambda^{'N} \right\|_{S_r} = \sup_{n \geq 1} \left(\frac{|\lambda|(N+n)}{|\lambda(N+n)-1|R} \right) \leq \rho_0,$$

which proves that for all $^t(b_N, b_{N+1}, \ldots) \in s_R$, (7) admits a unique solution in s_R. Hence $(C_\lambda^{'})^{-1} \in (s_R, s_R)$ and $(C_\lambda^{'})^{-1}$ belongs to S_R.

Proof of the proposition. For the study of equation

$$(C - \lambda I)X = B, \tag{9}$$

in s_R, $\lambda \notin J$, we need to define the matrices:

$$D' = \left(\frac{n}{1-n\lambda} \delta_{nm} \right), \quad Q = \begin{pmatrix} 1 & & \mathbf{0} & \\ -1 & 1 & & \\ & -1 & 1 & \\ \mathbf{0} & & . & . \end{pmatrix}; \tag{10}$$

to obtain the equality:

$$C_\lambda^{'} = D'(C - \lambda I)Q.$$

Letting $X = QX'$, (9) is equivalent to $C_\lambda^{'} X = D'B$. We deduct from the lemma that this equation admits in s_R a unique solution $X = Q(C_\lambda^{'})^{-1} D'B$. Hence $\lambda \notin \sigma_C$, which permits to conclude

Corollary 3. $\forall R > 1 \ \forall \lambda \notin J$:

$$\sup_{n \geq 2} \left(\frac{1}{|n\lambda - 1|R} + \frac{1}{|n\lambda - 1|\,|(n-1)\lambda - 1|R^2} + \cdots + \frac{1}{|n\lambda - 1| \cdots |2\lambda - 1| R^{n-1}} \right) < \infty.$$

This result comes from the calculation of $(C_\lambda^{'})^{-1}$ which gives:

$$\begin{bmatrix} 1 & & & & & \\ -c_2 & 1 & & & \mathbf{0} & \\ c_2 c_3 & -c_3 & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & -c_n c_{n-1} c_{n-2} & c_n c_{n-1} & -c_n & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

3.2. CASE WHERE $\lambda = \frac{1}{N}$, FOR A GIVEN INTEGER $N \geq 1$.

We study here the map $X \mapsto \left(C - \frac{1}{N} I\right) X$, from s_R into itself, with $R > 1$. It is necessary to define the matrix $\tilde{C}_N$ deduced from $C - \frac{1}{N} I$, by deleting the first N rows, and the first N columns. We have to consider, at first the case where N = 1, because it is different from the others.

3.2.1. CASE $\lambda = 1$.

We have the following result, where R is a real $> \frac{3}{2}$:

Proposition 4. For all $B^* = {}^t(0, b_2, b_3, \ldots) \in s_R$ there exist two sequences (γ_n) and (γ'_n) such that equation

$$(C - I)X = B^* \tag{11}$$

admits in s_R infinitely many solutions which can be written under the form:

$${}^tX = (u, \gamma_1 u + \gamma'_1, \ldots, \gamma_n u + \gamma'_n, \ldots)$$

for all complex number u.

Proof. Letting $x_1 = u$, (11) is equivalent to the system:

$$\begin{cases} \left(\frac{1}{2} - 1\right) x_2 = b_2 - \frac{1}{2} u, \\ \frac{1}{n} x_2 + \frac{1}{n} x_3 + \cdots + \left(\frac{1}{n} - 1\right) x_{n-1} = b_n - \frac{1}{n} u, \quad n = 3, 4, \ldots . \end{cases}$$

for all value of u. This one can be written $\tilde{C}_1 X = \tilde{B}^*_u$ ($\tilde{C}_1$ defined as above). Consider, now, the product:

$$\tilde{C}_1' = \tilde{D}\tilde{C}_1 Q,$$

with $\tilde{D} = \left(-\frac{n+1}{n}\delta_{nm} \right)$. Then we obtain:

$$\tilde{C}_1' = \begin{pmatrix} 1 & & & & \\ -\frac{3}{2} & 1 & & \mathbf{0} & \\ & -\frac{4}{3} & 1 & & \\ & & . & . & \\ & \mathbf{0} & & -\frac{n+1}{n} & 1 \\ & & & & . \quad . \end{pmatrix}.$$

Since we have

$$\left\| I - \tilde{C}_1' \right\|_{S_R} = \sup_{n \geq 2} \left(\frac{n+1}{nR} \right) < 1,$$

as $R > \frac{3}{2}$, equation $\tilde{C}_1' X = \tilde{B}_u^*$, admits for any u a unique solution $X = (\tilde{C}_1')^{-1} \tilde{B}_u^*$ in s_R. Call $c_{nm}^{\sim}$ the elements of $(\tilde{C}_1')^{-1}$ a simple calculation gives $\gamma_1 = b_2$, $\gamma_1' = -\frac{1}{2}$, and for $n \geq 2$:

$$\begin{cases} \gamma_n = b_{n+1} + \sum_{m=1}^{n-1} c_{nm}^{\sim} b_{m+1}, \\ \gamma_n' = \dfrac{1}{n+1} + \sum_{m=1}^{n-1} \dfrac{1}{m} c_{nm-1}^{\sim}; \end{cases}$$

which permits to conclusion.

3.2.2. CASE WHERE $\lambda = \frac{1}{N}$, $N \geq 2$.

We need a definition to formulate the following results.

Definition 5. One can associate to every N-1-tuple $b_1, \ldots, b_{N-1}$ of complex numbers, the linear forms ℓ_n, $n = 1, 2, \ldots, N-1$, defined by:

$$\ell_n(b_1, \ldots, b_n) = \sum_{s=1}^{n} \gamma_{ns} b_s,$$

where (γ_{ns}) is the sequence defined by:

$$\gamma_{ns} = \begin{cases} \dfrac{nN}{N-n} & \text{if} \quad s = n, \\ -\dfrac{(n-1)N}{N-(n-1)}(1+c_n) & \text{if} \quad s = n-1, \\ (-1)^{n-s}\dfrac{sN}{N-s}(1+c_n)\displaystyle\prod_{k=s+1}^{n-1} c_k & \text{if} \quad 1 \le s \le n-2 . \end{cases} \tag{12}$$

and we let:

$$u_N = \frac{1}{N}\sum_{n=1}^{N-1} \ell_n(b_1,\dots,b_n) .$$

We have the following result, where s is the space of all the complex sequences:

Proposition 6. $\forall\, B = {}^t(b_1,\dots,b_N,\dots) \in s$ with $b_N \neq u_N$ equation

$$\left(C - \frac{1}{N}I\right)X = B$$

does not admit a solution in s.

Proof. The N-th term of the diagonal of the matrix $\left(C - \frac{1}{N}I\right)$ being equal to 0, the determination of u_N depends on the solution $(x_1^0, x_2^0, \dots, x_{N-1}^0)$ of the finite linear system:

$$\begin{cases} \lambda_1 x_1 = b_1, \\ \dfrac{1}{n-1}x_1 + \cdots + \dfrac{1}{n-1}x_{n-2} + \lambda_{n-1}x_{n-1} = b_{n-1} \quad n = 3,4,\dots,N, \end{cases} \tag{13}$$

where $\lambda_n = \frac{1}{n} - \frac{1}{N}$. Now we search for an expression of this solution. The calculation of

$$C_N^* = D_N^*\left(C - \frac{1}{N}I\right)Q \tag{14}$$

where Q is defined in (10), and

$$D_N^* = \operatorname{diag}\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_{N-1}}, N, \frac{1}{\lambda_{N+1}}, \dots\right),$$

gives

$$C_N^* = \begin{pmatrix} 1 & & & & & & & \\ c_2 & 1 & & & & \mathbf{0} & & \\ & c_3 & 1 & & & & & \\ & & \cdot & \cdot & & & & \\ & & & c_{N-1} & 1 & & & \\ & & & & 1 & 0 & & \\ & \mathbf{0} & & & & c_{N+1} & 1 & \\ & & & & & & \cdot & \cdot \end{pmatrix}.$$

Letting ${}^t Y_N = (y_1, ..., y_{N-1}, 0, ...)$, we see that, using (6) where we call C'_N the matrix C'_λ :

$$Z_N = (C'_N)^{-1} Y_N \tag{15}$$

is the unique solution of the equation $C_N^* X = Y_N$. Using the series, as it is defined in (5), we can write:

$$Z_N = \sum_{k=0}^{\infty} (I - C'_N)^k Y_N$$

in the Banach space s_R, $(R > 1)$. Denote for every $k \geq 0$ $(I - C'_N)^k = (\alpha_{nm,k})_{n,m}$. A simple calculation gives

$$\alpha_{nm,k} = \begin{cases} (-1)^k \prod_{i=m+1}^{m+k} c_i \ , & \text{if} \quad n = m + k - 1, \quad \forall\ m \geq 1, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Hence, if z_n denotes the n-th component of Z_N, we obtain:

$$z_n = y_n + \sum_{s=1}^{n-1} (-1)^{n-s} \left(\prod_{k=s+1}^{n} c_k \right) y_s, \quad n = 1, 2, ..., N-1. \tag{16}$$

Let

$$\begin{cases} {}^t B_N = (b_1, b_2, ..., b_{N-1}, 0, ...), \\ Y_N = D_N^* B_N. \end{cases} \tag{17}$$

Then we obtain the solution of (13), calculating:

$$X_0 = \left(C - \frac{1}{N} I \right)^{-1} B_N,$$

where ${}^t X_0 = (x_1^0, x_2^0, ..., x_{N-1}^0, 0, ...)$. So, using (14), (15) and (17) we have

$$X_0 = (Q C_N'^{-1} D_N^*) B_N = Q Z_N,$$

which permits to obtain:

$$x_1^0 = z_1,\ x_n^0 = z_n - z_{n-1} \quad (2 \le n \le N). \tag{18}$$

Using (17), we get $y_n = \frac{n}{1-n\lambda} b_n$, which gives, replacing the expression of z_n (16) in (18):

$$x_n^0 = \sum_{s=1}^{n} \gamma_{ns} b_s \qquad n \ge 2,$$

with:

$$\gamma_{nn} = \frac{n}{1-n\lambda} = \frac{nN}{N-n}, \quad \gamma_{nn-1} = -\frac{(n-1)N}{N-(n-1)}(1+c_n)$$

and for $1 \le s \le n-2$, γ_{ns} has the form given by (12). One can conclude that if $b_N \ne \frac{1}{N}\sum_{n=1}^{N-1} \ell_n(b_1,\dots,b_n)$ equation $\left(C - \frac{1}{N}I\right)X = B$ does not admit a solution in the space s.

Let us see now, what happens when $b_N = u_N$.

Proposition 7. There exists a real $R_0 > \frac{N}{2} + 1$, such that $\forall\, B \in s_{R_0}$ verifying $b_N = u_N$, equation

$$\left(C - \frac{1}{N}I\right)X = B \tag{19}$$

admits infinitely many solutions in s_{R_0}, which can be written under the form:

$$X = {}^t(\ell_1(b_1), \ell_2(b_1, b_2), \dots, \ell_{N-1}(b_1,\dots,b_{N-1}), \upsilon, \gamma_1\upsilon + \gamma'_1, \dots, \gamma_n\upsilon + \gamma'_n, \dots)$$

for all υ, (γ_n) and (γ'_n) being two sequences depending on N and B.

Proof. Define here $\tilde{D}_N = \text{diag}\left(\frac{1}{\lambda_{N+1}}, \frac{1}{\lambda_{N+2}}, \dots\right)$ where $\lambda_n = \frac{1}{n} - \frac{1}{N}$. When $b_N = \frac{1}{N}\sum_{n=1}^{N-1} \ell_n(b_1,\dots,b_n)$, and $x_N = \upsilon$, is arbitrary, equation (19) is equivalent to:

$$\tilde{D}_N \tilde{C}_N X_N = B'_N(\upsilon);$$

$\tilde{C}_N$ being defined in 3.2.1, that is letting $\tilde{C} = \tilde{D}_N \tilde{C}_N$:

$$\tilde{C} = \begin{pmatrix} 1 & & & \mathbf{0} & \\ \frac{1}{(N+2)\lambda_{N+2}} & 1 & & & \\ . & . & . & & \\ \frac{1}{(N+n)\lambda_{N+n}} & . & \frac{1}{(N+n)\lambda_{N+n}} & 1 & \\ . & . & . & . & . \end{pmatrix},$$

and

$$\begin{cases} {}^tX_N = (x_{N+1}, x_{N+2}, \dots, x_{N+n}, \dots), \\ B'_N(\upsilon) = (b'_{N+1}(\upsilon), b'_{N+2}(\upsilon), \dots, b'_{N+n}(\upsilon), \dots), \end{cases}$$

with:

$$b'_{N+n}(\upsilon) = b'_{N+n} - \frac{\upsilon}{(N+n)\lambda_{N+n}} - \frac{1}{(N+n)\lambda_{N+n}} \sum_{n=1}^{N-1} \ell_n(b_1, \dots, b_n).$$

The calculation of $\tilde{C}Q = (\widetilde{c_{nm}})$ gives: $\tilde{c}_{nm} = 1 \quad \forall n; \tilde{c}_{nn-1} = \frac{N}{n} + 1 \quad \forall n \geq 2;$ and $\tilde{c}_{nm} = 0 \quad \forall m \neq n, n-1$. Since $R_0 > \frac{N}{2} + 1$, we have:

$$\left\| I - \tilde{C}Q \right\|_{s_{R_0}} = \left(\frac{N}{2} + 1 \right) \frac{1}{R_0} < 1,$$

then $\forall \upsilon, B \in s_{R_0}$ there exists a unique $X_N \in s_{R_0}$, such that: $\tilde{C}X_N = B'_N(\upsilon)$
Hence:

$$X_N = (\tilde{C})^{-1} B'_N(\upsilon) = {}^t(\gamma_1 \upsilon + \gamma'_1, \dots, \gamma_n \upsilon + \gamma'_n, \dots)$$

(γ_n) and (γ'_n) being two sequences obtained from $(\tilde{C})^{-1}$ and from $b_1, b_2, \dots, b_{N-1}, b_{N+1}, \dots$. see 3.2.1.

Remark 1. More precisely, we can determine the sequences (γ_n) and (γ'_n) by the following method: $(\tilde{C})^{-1} = Q(\tilde{C}Q)^{-1}$ where, (as in corollary 3),

$$(\tilde{C}Q)^{-1} = \begin{pmatrix} 1 & & \mathbf{0} & \\ -\alpha_2 & 1 & & \\ \alpha_2\alpha_3 & -\alpha_2 & 1 & \\ . & . & . & . \end{pmatrix}$$

with $\alpha_n = \frac{N}{n} + 1$. Then write $B'_N(\upsilon) = \beta_N + \upsilon\beta'_N$, with:

$$\begin{cases} \beta_N = \left(b'_{N+n} - \dfrac{1}{(N+n)\lambda_{N+n}} \displaystyle\sum_{n=1}^{N-1} \ell_n(b_1,\dots,b_n) \right)_{n\geq 1}, \\ \beta'_N = \left(-\dfrac{1}{(N+n)\lambda_{N+n}} \right)_{n\geq 1}. \end{cases}$$

Hence $(\gamma_n) = Q(\tilde{C}Q)^{-1}\beta_N$, and $(\gamma'_n) = Q(\tilde{C}Q)^{-1}\beta_N$, these calculations can be made easily.

Remark 2. The previous proposition shows that the image space $(C-\lambda I)(s_{R_0})$ is the set of the one-column matrices: ${}^t(b_1,\dots,b_{N-1},u_N,b_{N+1},\dots b_n,\dots)$, where $b_1,\dots,b_{N-1},b_{N+1},\dots,b_n,\dots$ are any complex numbers.

We deduct easily another result, see [10], always in the case where $\lambda = 1/N, N \geq 2, R_0$ being defined as in Proposition 7.

Corollary 8. The eigenvector, corresponding to the eigenvalue $\lambda = 1/N$, can be written under the form:

$${}^tV = \left(0,\dots,0,1,\binom{N}{1},\binom{N+1}{2},\dots,\binom{N+n}{n+1},\dots \right),$$

(1 being in the N-th position), and belongs to the space s_{R_0}.

In fact, $\left(C - \dfrac{1}{N} I \right) X = 0$ implies that $\ell_n(b_1,\dots,b_n) = 0$, $\forall n = 1,2,\dots N-1$, and the system:

$$\begin{cases} x_{N+1} = \upsilon, \\ \dfrac{1}{N+n}(x_{N+1} + \cdots + x_{N+n-1}) - \dfrac{n-1}{N(N+n)} x_{N+n} = 0, \quad n = 2,3,\dots \end{cases}$$

is equivalent to the matrix equation:

$$\begin{pmatrix} -\frac{1}{N} & & \mathbf{0} & \\ 1 & -\frac{2}{N} & & \\ 1 & 1 & -\frac{3}{N} & \\ . & . & . & . \end{pmatrix}\begin{pmatrix} x_{N+2} \\ x_{N+3} \\ . \\ . \end{pmatrix} = -\begin{pmatrix} 1 \\ (N+2)\upsilon \\ . \\ . \end{pmatrix}.$$

We deduct, taking $x_{N+1} = \upsilon = 1$, that $x_{N+2} = \binom{N}{1}$; the relation: $x_{N+2} - \frac{2}{N} x_{N+3} = -(N+2)$ implies that $x_{N+3} = \binom{N+1}{2}$ and so on... By induction, using the well known property of $\Gamma_n^p = \binom{n+p-1}{p}$ number of combinations of n things, p at a time, when repetitions are allowed, we have:

$$x_{N+n} = \frac{N}{n-1}\left[1 + \binom{N}{1} + \binom{N+1}{2} + \cdots + \binom{N+n-3}{n-2}\right] = \binom{N+n-2}{n-1}$$

which achieves the proof.

An analogous calculation can be done for N=1.

Remark 3. We deduct from the previous result that $\binom{N+n}{n} = O(R_0^n)$ $(n \to \infty)$ since $V \in s_{R_0}$.

4. VARIATION OF AN ELEMENT IN THE MATRIX C.

Until here, we have considered matrices deducted from C, by replacing all the elements of the main diagonal a_{nn} by $a_{nn} - \lambda$; now we study the case where only one element of the Cesàro matrix is changed. For this, we consider a given row μ and a given column ν and denote by a the term $a_{\mu\nu}$ of the matrix C. B being given, we study what becomes the solution of equation CX=B, when a is replaced by another element a' in the matrix C ; C' will denote this new matrix. To know if the new equation, that is $C'X = B$, admits or not a solution; we need the following result:

Proposition 9. Equation $C'X = B$, admits in s a unique solution in the following cases:

i) $\mu = \nu$, and $\alpha \neq 0$,

ii) $\mu = \nu - 1$ and $\alpha \neq 1/\mu$,

iii) for all value of α, if μ different from ν and $\nu - 1$.

Proof. i) is obvious, because if C' has a zero element on the main diagonal it is not invertible; for ii) $\alpha = 1/\mu$, two rows have their corresponding elements proportional which implies, too, that C' is not invertible. For iii) it is easy to see that the new system has always one solution if $\nu < \mu$, since all the diagonal elements are different from 0. If $\nu > \mu + 1$ consider the system:

$$\begin{cases} \frac{1}{n}\sum_{m=1}^{n} x_m = b_n, \quad n = 1,2,\ldots,\mu-1,\mu+1,\ldots,\nu, \\ \\ \frac{1}{\mu}\sum_{m=1}^{\mu} x_m + \alpha x_\nu = b_\mu. \end{cases} \tag{20}$$

A simple calculation proves that the determinant of the coefficients of the variables is equal to $1/\nu!$ and different from 0 for all $\alpha \neq 0$. Call $x_1^0, x_2^0, \ldots, x_\nu^0$ the coordinates of the solution of (20), equation $C'X = B$ being equivalent to $\tilde{C}_\nu X_\nu = B_\nu$, $\tilde{C}_\nu$ defined in 3.2, and:

$${}^tX_\nu = (x_{\nu+1},\ldots) \quad {}^tB_\nu = \left(b_{\nu+1} - \frac{1}{\nu+1}\sum_{i=1}^{\nu} x_i^0, \ldots, b_n - \frac{1}{n}\sum_{i=1}^{\nu} x_i^0, \ldots \right),$$

Admits a unique solution, which permits to calculation.

B being given, denote by $X_a(\mu,\nu)$, (or X_a) the solution of CX=B and by $X_\alpha(\mu,\nu)$, (or X_α) the solution of $C'X = B$ whenever they exist. As we shall see the difference $X_a(\mu,\nu) - X_\alpha(\mu,\nu)$ is a vector having only two elements different from 0, we can then write: $d_\alpha(\mu,\nu) = \|X_a(\mu,\nu) - X_\alpha(\mu,\nu)\|_{s_1}$ so we have the following result:

Proposition 10. For a given matrix $B \in s_1$ and a given real α we have:

i) if $\mu = \nu$, or $\mu = \nu - 1$ and $\alpha \neq 0$:

$$d_\alpha(\mu,\nu) = |\alpha| O(\mu^2), \tag{21}$$

as μ tends to infinity;

ii) if $\mu + 1 < \nu$:

$$d_{\alpha'}(\mu,\nu)=|\alpha'|O(1), \tag{22}$$

as ν tends to infinity, μ being fixed;

iii) if $\mu\geq\nu+1$:

$$d_{\alpha'}(\mu,\nu)=|\alpha'|O(\mu), \tag{23}$$

as μ tends to infinity, for fixed ν.

Proof. i) If $\mu=\nu$, letting

$$\Delta_\mu=\operatorname{diag}(1,2,\ldots,\mu-1,\mu\alpha',\mu+1,\ldots)$$

we get $\Delta_\mu C'Q=(\alpha_{nn})$, with $\alpha_{nn}=1$, $\forall n\neq\mu$; $\alpha_{\mu\mu}=\mu\alpha'$; $\alpha_{\mu\mu-1}=1-\mu\alpha'$; and $\alpha_{nm}=0$ for the other values of n, m. Since we have

$$X_{\alpha'}(\mu,\nu)=Q(\Delta_\mu C'Q)^{-1}\Delta_\mu B$$

and $(\Delta_\mu C'Q)^{-1}=(\alpha'_{nm})$, with $\alpha'_{nn}=1$, $\forall n\neq\mu$; $\alpha'_{\mu\mu}=1/\mu\alpha'$; $\alpha'_{\mu\mu-1}=-1+\mu\alpha'$ the other elements being equal to 0; we deduct that the n-th coordinate ξ_n of $X_{\alpha'}(\mu,\nu)$ is

$$\xi_1=b_1,\ \xi_n=-(n-1)b_{n-1}+nb_n,\ \text{if}\ \ n=2,\ldots,\mu-1,\mu+2,\ldots;$$

and for $n=\mu$, or $\mu+1$:

$$\begin{cases}\xi_\mu=-(\mu-1)b_{\mu-1}+(\mu\alpha'-1)(\mu-1)b_{\mu-1}+\dfrac{1}{\alpha'}b_\mu,\\[2ex] \xi_{\mu+1}=-(\mu\alpha'-1)(\mu-1)b_{\mu-1}-\dfrac{1}{\alpha'}b_\mu+(\mu+1)b_{\mu+1}.\end{cases}$$

The calculation of $X_{\alpha'}(\mu,\nu)-X_\alpha(\mu,\nu)=(\sigma_n)$, gives:

$$\begin{cases}\sigma_\mu=(1-\mu\alpha')\left[(\mu-1)b_{\mu-1}+\dfrac{1}{\alpha'}b_\mu\right],\\[2ex] \sigma_{\mu+1}=(1-\mu\alpha')\left[(\mu-1)b_{\mu-1}-\dfrac{1}{\alpha'}b_\mu\right],\end{cases} \tag{24}$$

and $\sigma_n=0$, $\forall n\neq\mu$, $\mu+1$. We see that $(b_n)\in s_1$ implies that

$$\|X_{\alpha'}(\mu,\nu)-X_\alpha(\mu,\nu)\|_{s_1}=\sup(|\sigma_\mu|,|\sigma_{\mu+1}|)=|\alpha'|O(\mu^2)\ \ (\mu\to\infty).$$

If $\mu=\nu-1$, Δ_μ defined above, is replaced by $\Delta'_\mu=(n\delta_{nm})$, then $\Delta'_\mu C'Q=(\beta_{nm})$, with $\beta_{nm}=1$, $\forall n\neq\mu$; $\beta_{\mu\mu}=1-\alpha'\mu$; $\beta_{\mu\mu+1}=\alpha'\mu$; and $\beta_{nm}=0$ for the other values of n,m; the calculation gives:

$$\begin{cases}\sigma_\mu = \dfrac{\alpha'\mu^2}{\alpha'\mu - 1} b_\mu + \alpha'\mu(\mu+1)b_{\mu+1}, \\ \sigma_{\mu+1} = -\sigma_\mu,\end{cases} \tag{25}$$

which proves (21).

ii) If $\mu+1<\nu$ an analogous calculation gives:

$$\begin{cases}\sigma_\mu = -\alpha'\mu[\nu b_\nu - (\nu-1)b_{\nu-1}], \\ \sigma_{\mu+1} = \alpha'\mu[\nu b_\nu - (\nu-1)b_{\nu-1}],\end{cases}$$

σ_n being equal to 0, for every $n \neq \mu$, $\mu+1$. Hence, for a fixed value of μ one gets (22).

iii) If $\mu \geq \nu+1$, we get as nonvanishing terms:

$$\begin{cases}\sigma_\mu = (1-\alpha'\mu)[\nu b_\nu - (\nu-1)b_{\nu-1}], \\ \sigma_{\mu+1} = (1-\alpha'\mu)[\nu b_\nu + (\nu-1)b_{\nu-1}],\end{cases} \tag{26}$$

which permits to obtain (23).

Remark 4. In the case $\mu = \nu - 1$, if $\alpha' = 1/\mu$ we see that, B being given such that $\mu b_\mu \neq (\mu+1) b_{\mu+1}$ equation $C'X = B$ has no solution, (since the rows of indices μ, and $\mu+1$ are proportional). Suppose now that we replace α' by $b' = \alpha' + \varepsilon$, ε being any number different from 0, in the matrix C'; calling C'' the new matrix so defined, we deduct that equation $C''X = B$ admits one solution $X_{b'}$, such that:

$$\|X_{b'}(\mu,\nu)\|_{s_1} = \varepsilon\mu^2 O(1) \quad (\mu \to \infty).$$

So we see that a slight variation of an element can imply a large variation of the solution if we consider a big value of the index μ of the row, in which we do the substitution.

One studies, now, the case where we do a second variation of an element, always in the same row μ and in the same column ν, so $a = a_{\mu\nu}$ is replaced by b'; as above we denote by $X_{b'}$ the corresponding solution, we get then the following results:

Corollary 11. If $\mu = \nu$ or $\nu - 1$ then:

$$\|X_{\alpha'} - X_{b'}\|_{s_1} = |\alpha' - b'|\mu^2 O(1) \quad (\mu \to \infty); \tag{27}$$

if $\mu \geq \nu + 1$:

$$\|X_{a'} - X_{b'}\|_{s_1} = |a' - b'|\mu\, O(1) \quad (\mu \to \infty). \tag{28}$$

Proof. It is enough to write $X_{a'} - X_{b'} = X_{a'} - X_a + X_a - X_{b'}$; so, if $\mu = \nu$ using (24) we get:

$$\|X_{a'} - X_{b'}\|_{s_1} = |a' - b'| \sup\left(\left|\mu(\mu-1)b_{\mu-1} + \frac{1}{a'b'} b_\mu\right|, \left|\mu(\mu-1)b_{\mu-1} - \frac{1}{a'b'} b_\mu\right| \right)$$

which proves (27). If $\mu = \nu - 1$, letting $X_{a'} - X_{b'} = (\sigma'_n)$ one deducts from (25), that; $\sigma'_n = 0$, when $n \neq \mu$, $\mu + 1$ and:

$$|\sigma'_\mu| = |\sigma'_{\mu+1}| = |a' - b'|\mu \left| \frac{\mu}{(1 - a'\mu)(1 - b'\mu)} b_\mu + (\mu + 1)\, b_{\mu+1} \right|$$

which implies (27). At last, for $\mu \geq \nu + 1$, we deduct from (26):

$$\|X_{a'} - X_{b'}\|_{s_1} = |a' - b'|\mu \sup(|(\nu-1)b_{\nu-1} - \nu b_\nu|, |(\nu-1)b_{\nu-1} + \nu b_\nu|),$$

which proves (28).

We deduct from the preceding corollary:

Corollary 12. If $\mu = \nu$, or $\nu - 1$ and $b' = a' + \tau$ for a given integer τ, then:

$$\|X_{a'} - X_{b'}\|_{s_1} = \tau\mu^2 O(1) \quad (a' \to \infty).$$

Remark 5. μ_1 and ν_1 being two integers, we can define, using the same notations, the solution $X_{b'}(\mu_1, \nu_1)$ of equation $C'_1 X = B$, (where C'_1 is now the matrix deducted from C, replacing $a = a_{\mu_1 \nu_1}$ by b') for a given B. Suppose, for instance, that $\nu + 2 \leq \mu + 1 < \mu_1$ and $\nu_1 < \mu_1$. Using (26), let

$$\begin{cases} \sigma_\mu = \sigma(\mu, \nu, a') = (1 - a'\mu)\,[\nu b_\nu - (\nu - 1) b_{\nu-1}], \\ \sigma_{\mu+1} = \sigma'(\mu, \nu, a') = (1 - a'\mu)\,[\nu b_\nu - (\nu - 1) b_{\nu-1}]. \end{cases}$$

Then we deduct:

$$\|X_{a'}(\mu, \nu) - X_{b'}(\mu_1, \nu_1)\|_{s_1} = \sup(|\sigma(\mu, \nu, a')|, |\sigma'(\mu, \nu, a')|, |\sigma(\mu_1, \nu_1, b')|, |\sigma'(\mu_1, \nu_1, b')|)$$

So one can do estimations, and obtain according to the values of μ, ν, μ_1, ν_1 analogous results to the preceding.

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