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LORENTZIAN CIRCLE AS A LIE GROUP AND A C^{∞} ACTION ON LORENTZ SPACES OF TWO DIMENSION

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ABSTRACT

We define a product on Lorentz circle that is similar to Lorentzian inner product. This product is expanded to Lorents space of two dimension. So we have an C^{∞} action of Lorentz circle on Lorents space of two dimension. It is noted that this action provides some isometrics of L^2.

1. INTRODUCTION

A Lie group G is a group which has the structure of a differentiable manifold and for which the group function

$$\Theta : G\times G \longrightarrow G$$

defined by $\Theta(g_1, g_2) = g_1 g_2$ is differentiable. Given an element a of a Lie group G, the function $L: G \to G$ defined $g \to ag$ is called left translation. A Lie group G is said to act on a differentiable manifold M as Lie transformation group if we are given a global surjection

$$\phi : G\times M \to M$$

which is differentiable such that if $g, h \in G$ and $m \in M$

$$\phi(g, \phi(h, m)) = \phi(gh, m)$$

G is said act transitively on M if, given any two points $m_1, m_2 \in M$ there is an element $g \in G$ such that $m_2 = gm_1$.

By a transformation of a manifold M, we mean a differomorphism of M onto itself. A group G is said to act on M as a transformation group. If there is a global function

$\phi : G\times M \to M$

such that

i) the function ϕ_g, defined for any given $g\in G$ by $m \to \phi(g,m)$ is a transformation of M,

ii) if $g,h\in G, \phi_g \circ \phi_h = \phi_{gh}$.

Suppose that e is the unit element of G, then ϕ_e is the identity element on M, for if $m\in M$ and $m' = (\phi_e - 1)_m$ The group is said to act effectively on M if e is the only element of G such that $\phi_g m = m$ for all $m \in M$. It is said to act freely on M if e is the only element of G such that $\phi_g m = m$ for some M.

A transformation group G acting on a manifold M sets up an equivalence relation on M. The equivalence class containing a point m is the range of the function $\phi_m : G \to M$ and we call it the orbit of m [1].

We will denote Lorenzian circle on L^2 by L_1^1. In this study, it was shown that L_1^1 is a Lie group with a binary operation defined on L^1 as hypercylindrical product defined by [3]. A C^∞- action on L^2 of this Lie group is defined and some properties of it were given. Finally, with the help of this C^∞-action some isometries of L^2 were obtained .

1.1 LORENTZ MANIFOLDS, LORENTZ VECTOR SPACES AND LORENTZ CIRCLE

A metric tensor g on a differentiable manifold M is a symmetric nondegenerate (0,2) tensor field on M of constant index. A semi-Riemannian manifold is a differentiable manifold M furnished with a metric tensor g. Thus a semi-Riemannian manifold is an ordered pair (M,g).

The common value ν of index g_p on a semi-Riemannian manifold M is called the index of M satisfying $0 \le \nu \le n = \dim M$. If $\nu = 0$, M is a Riemannian manifold since each g_p is an (positive definite) inner product on $T_p(M)$ If $\nu = 1$ and $n \le 2, M$ is Lorentz manifold.

In this study, v will take the values 1 and 2, and L^2 will denote a Lorentz manifold of two dimension. So, the metric tensor is defined by

$$< X, Y >_L = x_1 y_1 - x_2 y_2$$

or

$$< X, Y >_L = < X, Y_s >_E$$

where Y_s is symmetry of Y according to the x-axes and $X = (x_1, x_2)$, $Y = (y_1, y_2)$ [4].

2 LIE GROUP STRUCTURE OF L_1^1

We define a binary operation on L_1^1 by

$$\odot : L_1^1 \times L_1^1 \longrightarrow L_1^1$$

$$\odot (X, Y) = (\langle X, Y \rangle_E, \langle X, Y_S \rangle_E)$$

where $\langle , \rangle_E$ is Euclidean inner product and Y_S is symmetry of $Y \in R^2$ according to the straight line $y = x$. We have, the following

Theorem 1. The system $(L_1^1, \odot)$ is a commutative group.

Proof. For all $X, Y \in L_1^1$,

1. $\odot(X,Y) \in L_1^1$,
2. $\odot(X,Y) = (x_1 y_1 + x_2 y_2, x_1 y_2 + x_2 y_1)$
   $= (x_1 y_1 + x_2 y_2, y_1 x_2 + y_2 x_1)$
   $= \odot(Y, X)$

3. e=(1,0) is the identity element
4. The inverse element of $X = (x_1, x_2)$ is $(x_1 - x_2)$.

By Theorem 1, L_1^1 becomes a Lie group since $L_1^1 \subset R^2$ is a differentiable submanifold, and the symmetry function and the inner product are differentiable functions.

For $r \in R^+$, we define the set L_r^1 as

$$L_r^1 = \{(x, y) \in L^2 \,|\, x^2 - y^2 = r^2\}$$

and action $\theta : L_1^1 \times L_r^1 \to L_r^1, \theta((x_1, x_2), (y_1, y_2)) = (x_1 y_1 + x_2 y_2, x_1 y_2 + x_2 y_1)$.

Theorem 2. L_1^1 acts transitively on L_r^1.

Proof. For $p, q \in L_r^1$ we can define an element $X = (x_1, x_2)$ where

$$x_1 = \frac{<p,q>}{r^2} \quad \text{and} \quad x_2 = \frac{\langle p, q_s \rangle}{r^2}.$$

Then, it is clear that $X \in L_1^1$ and

$$\theta(X, Q) = P$$

This completes the proof.

For $X \in L_r^1, (\theta)_x$ the orbit of X under the action θ is L_r^1.

Theorem 3. L_1^1 acts effectively on L_1^1.

Proof. We have to show that for all $m \in L_1^1$ the equation $\theta(g, m) = m$ is satisfied only for g=e. In fact

$$\theta(g, m) = m \;\Rightarrow\; g_1 m_1 + g_2 m_2 = m_1$$
$$g_1 m_2 + g_2 m_1 = m_2$$
$$\Rightarrow\; g_1 = \frac{\det\begin{bmatrix} m_1 & m_2 \\ m_2 & m_1 \end{bmatrix}}{r^2} = 1$$
$$g_2 = \frac{\det\begin{bmatrix} m_1 & m_1 \\ m_2 & m_2 \end{bmatrix}}{r^2} = 0$$

so $g = e$.

3. THE SET $\tilde{L}_r^1$

We define the set $\tilde{L}_r^1$ as

$$\tilde{L}_r^1 = \{(x, y) \mid y^2 - x^2 = r^2\}$$

so we can define an action

$$\overline{\odot} : \tilde{L}_r^1 \times \tilde{L}_r^1 \longrightarrow \tilde{L}_r^1$$
$$\overline{\odot}(X, Y) = (\langle X, Y \rangle_E, \langle X, Y_S \rangle_E)$$

In this case, we evidently have;

1. $(\tilde{L}_1^1, \overline{\odot})$ is a commutative group
2. $\tilde{L}_r^1$ is a Lie group

3. $\tilde{L}_1^1$ acts on $\tilde{L}_r^1$ as a Lie transformation group with the function $\tilde{\theta}$ defined by
$\tilde{\theta}(g,X) = (\langle g,X\rangle_E, \langle g,X_S\rangle_E)$
4. $\tilde{L}_1^1$ acts transitively on $\tilde{L}_r^1$
5. $\tilde{L}_1^1$ acts effectively on $\tilde{L}_r^1$
6. $(\tilde{\theta})_X = \tilde{L}_r^1$, where $X \in \tilde{L}_r^1$

3.1. AN ACTION ON L^2

For all $X = (x_1, x_2) \in L^2$, we have
$x_1^2 - x_2^2 = r^2$ or $x_2^2 - x_1^2 = r^2$
where $r \in \mathbf{R}^+ \cup \{0\}$. So, we write

$$(\bigcup_r L_r^1) \cup (\bigcup_r \tilde{L}_r^1) \supseteq L^2.$$

Then, we conclude

Theorem 4. L_1^1 acts on L^2 as a Lie transformation group with the function θ' defined by

$$\theta'(X,Y) = \begin{cases} \theta(X,Y), & \text{if} \quad y_1 \geq y_2 \\ \tilde{\theta}(X,Y), & \text{if} \quad y_1 < y_2 \end{cases}$$

For all $p \in L^2$, the orbit of $p = (p_1, p_2)$ under θ' is

$$(L_1^1)_{(p)} = \begin{cases} L_r^1, & p_1 \geq p_2 \\ \tilde{L}_r^1, & p_1 < p_2 \end{cases}.$$

Also, for all $g \in L^2$ the mappings $\theta' : L^2 \longrightarrow L^2$ defined by
$\theta'_g(X) = \theta'(g,X)$
are diffeomorphisms.

Theorem 5. The mappings θ'_g are isometrics of L^2.

Proof. Let $X = (x_1, x_2)$, $Y = (y_1, y_2) \in L^2$ and $g = (g_1, g_2) \in L_1^1$. Thus
$d_L(\theta'_g(X), \theta'_g(Y))^2 = d_L[(\langle g,X\rangle, \langle g,X_S\rangle), (\langle g,Y\rangle, \langle g,Y_S\rangle)]^2$

$$
\begin{aligned}
&= (\langle g, X\rangle - \langle g, Y\rangle)^2 - (\langle g, X_S\rangle, \langle g, Y_S\rangle)^2 \\
&= \langle g, X - Y\rangle^2 - \langle g, X_S - Y_S\rangle^2 \\
&= (g_1(x_1 - y_1) + g_2(x_2 - y_2))^2 - (g_1(x_2 - y_2) + g_2(x_1 - y_1))^2 \\
&= g_1^2(x_1 - y_1)^2 + g_2^2(x_2 - y_2)^2 + 2g_1g_2(x_1 - y_1)(x_2 - y_2) \\
&\quad - g_1^2(x_2 - y_2)^2 - g_2^2(x_1 - y_1)^2 - 2g_1g_2(x_2 - y_2)(x_1 - y_1) \\
&= (g_1^2 - g_2^2)(x_1 - y_1)^2 - (g_1^2 - g_2^2)(x_2 - y_2)^2 \\
&= (x_1 - y_1)^2 - (x_2 - y_2)^2 \\
&= d_L(X, Y)^2
\end{aligned}
$$

where $x_1 < x_2$, $y_1 < y_2$. All other possibilities, which are $x_1 \geq x_2$, $y_1 < y_2$ or $x_1 \geq x_2$, $y_1 \geq y_2$ or $x_1 < x_2$, $y_1 \geq y_2$ can be verified as above.

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