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ON THE NEWTON INTERPOLATING SERIES OF ENTIRE FUNCTIONS

D. KUMAR [(1)] *and R.K. YADAV* [(2)]

[(1)] *Department of Mathematics, D.S.M. Degree College, Kanth-244 501, (Moradabad) U.P., INDIA*

[(2)] *Department of Mathematics, Hindu College, Moradabad-244001 U.P., INDIA*

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Abstract. This paper deals with coefficients characterization of lower (p,q)-order and generalized lower (p,q)-type of Newton's interpolating series of entire functions. Finally, some inequalities connecting generalized (p,q)-type and generalized lower (p,q)-type of interpolating series have been derived.

1. INTRODUCTION

Let $z_1, z_2, ..., z_n, ...,$ where z_n, s need not be all distinct be a bounded sequence of complex numbers and let f be analytic at the points z_n, $n = 1,2,...,$ set

$$w_0(z) = 1$$

$$w_n(z) = (z - z_1)(z - z_2),...,(z - z_n), \ n = 1,2,..., \tag{1.1}$$

A series of the form

$$\sum_{k=0}^{\infty} C_k W_k(z), \tag{1.2}$$

where $P_n(z) = \sum_{k=0}^{n} C_k W_k(z)$, is a polynomial of degree n which interpolates f in the points $z_1, z_2, ..., z_{n+1}$, is called Newton's interpolating series of f. It is well known [6], [7] that if the points z_n 's are chosen interior to a domain D within which f is analytic, it does not necessarily follow that $\lim_{n\to\infty} P_n(z) = f(z)$ for every $z \in D$. However, if f is an entire function, the series (1.2) converges uniformly on every compact subset of the complex-plane to f and the coefficients C_n are given by

$$C_n = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{w_{n+1}(t)}dt, \qquad n = 0,1,2,..., \tag{1.3}$$

where Γ is any contour which contains in its interior all the points z_n. It is clear that Newton's series of an entire function reduces to its Taylor Series about $z = 0$ if $z_n = 0$ for $n = 1,2,...$.

Definition 1. An entire function f(z) is said to be of (p,q)-order $\rho(p,q)$ and lower (p,q) order $\lambda(p,q)$ if it is of index-pair (p,q) such that

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} M(r)}{\log^{[q]} r} = \frac{\rho(p,q)}{\lambda(p,q)},$$

and the function f(z) having (p,q)-order $\rho(p,q)$ $(b < \rho(p,q) < \infty)$ is said to be of (p,q)-type T(p,q) and lower (p,q)-type t(p,q) if

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} M(r)}{\left(\log^{[q-1]} r\right)^{\rho}} = \frac{T(p,q)}{t(p,q)}, \quad 0 \le t(p,q) \le T(p,q) < \infty ,$$

where $b = 1$ if $p = q$, $b = 0$ if $p > q$.

Kasana [3] has extended the idea of proximate order to entire function with (p,q) growth as follows.

Definition 2. A positive function $\rho_{(p,q)}(r)$ defined on $[r_0, \infty)$, $r_0 > \exp^{[q-1]} 1$, is said to be a proximate order of an entire function with index-pair (p,q) if

(i) $\rho_{[p,q]}(r) \to \rho(p,q)$ as $r \to \infty$; $b < \rho(p,q) < \infty$,

(ii) $\Delta_{[q]}(r)\, \rho'_{[p,q]}(r) \to 0$ as $r \to \infty$; $\rho'_{[p,q]}(r)$ denotes the derivative of $\rho_{p,q}(r)$.

It is known that $(\log^{[q-1]} r)^{\rho_{(p,q)}(r)-A}$ is a monotonically increasing function of r for $r > r_0$. Hence we can define the function $\phi(x)$ to be the unique solution of the equation,

$$x = (\log^{[q-1]} r)^{\rho_{(p,q)}(r)-A} \Leftrightarrow \phi(x) = \log^{[q-1]} r \tag{1.4}$$

where A=1 if (p,q)=(2,2) and A=0, otherwise.

Definition 3. Let f(z) be an entire function of (p,q)-order $\rho(p,q)(b < \rho(p,) < \infty)$ such that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^{\rho_{(p,q)}(r)}} = \frac{T^*(p,q)}{t^*(p,q)}, \quad 0 \le t^*(p,q) \le T^*(p,q) \le \infty .$$

Let f be an entire function. Here we can write

$$f(z) = \sum_{n=0}^{\infty} C_n W_n(z) . \tag{1.5}$$

Winiarski [8] connected classical order and type of an entire function f in terms of the coefficients of the expansion of f into a Newton's interpolating series. These results have been extended to (p,q) scale introduced by Junaja et.al. ([1],[2]), by Rizvi [5].It has been noticed that these authors fail to compare the coefficients of those Newton's interpolating series which have same positive finite order but their types are infinity. For the view point of including this important class of entire Newton's interpolating series we shall utilise the concept of proximate order.

The text has been divided into two parts, Section 1, consists of introductory exposition of the topic and Section 2, includes the theorem which characterize the lower (p,q) –order in terms of the ratio of coefficients of the expansion of f into a Newton's interpolating series. Finally, some inequalities connecting generalized (p,q) type and generalized lower (p,q)-type of interpolating series in terms of these coefficients have been drived.

Before discussing the main results it will be justified to introduce with the concept of (p,q) –scale $p \ge q \ge 1$, and certain notations which will be frequently used in the text:

$$\exp^{[m]} x = \log^{[-m]} x = \exp(\exp^{[m-1]} x = \log(\log^{[m-1]} x), m = +1, +2, \ldots,$$

$$\Delta_{[r]}(x) = \prod_{i=0}^{r} \log^{[i]} x \text{ for } r = 0,1,\ldots,$$

$$P_\chi[L(p,q)] = \begin{cases} L(p,q) & \text{if} \quad q < p < \infty, \\ \chi + L(p,q) & \text{if} \quad p = q = 2, \\ \max(1, L(p,q)) & \text{if} \quad 3 \le p = q, \\ \infty & \text{if} \quad p = q = \infty \end{cases}$$

$$\gamma(p,q) = \begin{cases} (\rho(2,2) - 1)^{\rho(2,2) - 1/(\rho(2,2))^{\rho(2,2)}} & \text{if} \quad (p,q) = (2,2) \\ 1/e\rho(2,1) & \text{if} \quad (p,q) = (2,1) \\ 1 & \text{otherwise.} \end{cases}$$

Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ be an entire function. We set $M(r,f)=\max_{|z|=r}|f(z)|$; $M(r,f)$ is called the maximum modulus of f(z) on the circle $|z|=r$

If the quantity t*(p,q) is different from zero and infinite then $\rho_{p,q}(r)$ is said to be the proximate order of a given entire function f(z) and t*(p,q) (p,q) as its generalized lower (p,q)-type.

2. MAIN RESULTS

Theorem 1. Let $f(z)=\sum_{n=0}^{\infty}C_n W_n(Z)$ be an entire function with index-pair (p,q), lower (p,q)-order $\lambda(p,q)$ $(b<\lambda(p,q)<\infty)$ and generalized lower (p,q)-type t*(p,q), then there exists an entire function g(z) $g(z)=\sum_{n=0}^{\infty}|C_n|z^n$ such that

$$\lambda(p,q,f)=\lambda(p,q,g) \text{ and } t^*(p,q,f)=t^*(p,q,g). \tag{1.6}$$

Proof. We have for $|z|=r$ $(r>L=\sup_{n\geq 0}|z_n|)$.

$$|W_n(z)|\geq\left(1-\frac{L}{r}\right)^n r^n \text{ , } n\geq 1.$$

Using (1.3) we have

$$|C_n|\leq\left(\frac{r}{r-L}\right)^{n+1}\frac{M(r,f)}{r^n}, n\geq 1 \text{ and } r>L.$$

Hence

$$\max_{n\geq 1}|C_n|r^n=\mu(r;g)\leq\left(\frac{r}{r-L}\right)^{n+1}M(r,f) \tag{1.7}$$

where

$$g(z)=\sum_{n=0}^{\infty}|C_n|z^n. \tag{1.8}$$

Again it is easy to find for any $\varepsilon>0$.

$$M(r,f)<M(re^{\varepsilon};g) \tag{1.9}$$

for all large r holds.

Now using the inequalities (1.7),(1.9) in Definitions 1 and 3, we get the required results.

Theorem 2. Let $f(z)=\sum_{n=0}^{\infty} C_n W_n(Z)$. Then f(z) be an entire function of lower (p,q)-order $\lambda(p,q)(b<\lambda(p,q)<\infty)$ if and only if, for $(p,q)\neq(2,2)$,

$$\lambda(p,q)=\max_{\{n_k\}}\lfloor P_\chi(l(p,q))\rfloor \tag{1.10}$$

and

$$\lambda(p,q)=\max_{\{n_k\}}\lfloor P_\chi(l^*(p,q))\rfloor \tag{1.11}$$

where

$$\ell(p,q)=\liminf_{k\to\infty}\frac{\log^{[p-1]} n_{k-1}}{\log^{[q-1]}\left|C_{n_k}\right|^{-1/n_k}}$$

and

$$\ell^*(p,q)=\liminf_{k\to\infty}\frac{\log^{[p-1]} n_{k-1}}{\log^{[q-1]}\left(\frac{1}{n_k-n_{k-1}}\log\left|\frac{C_{n_{k-1}}}{C_{n_k}}\right|\right)}.$$

Such that

$$\chi\equiv\chi\{n_k\}=\liminf_{k\leftarrow\infty}\frac{\log n_{k-1}}{\log n_k}$$

further (1.10) and (1.11) hold for (p,q)=(2,2) also provided $\{n_k\}$ be the sequence of principal indices such that

$\log n_{k-1}\cong\log n_k$ as $k\to\infty$.

Proof. By Theorem 1, f(z) and g(z) have the same lower (p,q)-order. Applying Theorem 2 by Juneja et al. [1, pp 62] to the function $g(z)=\sum_{n=0}^{\infty}\left|C_n\right|Z^n$, Theorem 2 follows.

Theorem 3. Let $f(z)=\sum_{n=0}^{\infty}\left|C_n\right|W_n(z)$. Then f(z) be an entire function of (p,q)-order $\rho(p,q)$ $(b<(p,q)<\infty)$ and generalized lower (p,q)-type $t^*(p,q)$ $(0<t^*(p,q)<\infty)$ if and only if

$$t^*(p,q)=\max_{\{m_k\}}\left\{\liminf_{K\to\infty}\left[\frac{\phi(\log^{[p-2]} m_{k-1})}{\log^{[q-1]}\left|C_{m_k}\right|^{-1/m_k}}\right]^{\rho(p,q)}\right\},\ p\geq 3, \tag{1.12}$$

and further, if the sequence of principal of principal indices $\{n_k\}$ satisfies $n_{k-1} \cong$ as $k \to \infty$, then for p=2,

$$\frac{t^*(2,q)}{\gamma(2,q)} = \max_{\{m_k\}}\left\{\liminf_{k\to\infty}\left[\frac{\phi(m_{k-1})}{\log^{[A]}|C_{m_k}|^{-1/m_k}}\right]^{\rho(2,q)-A}\right\}, \tag{1.13}$$

where maximum is taken over all increasing sequence of positive integers, A is defined as in (1.4).

Proof. To prove this theorem we apply Theorem 2 by Kasana et. al., [4] to the function $g(z) = \sum_{n=0}^{\infty}|C_n|z^n$. and the resulting characterization of $t^*(p,q,g)$ in terms of C_n and the relation $t^*(p,q,f) = t^*(p,q,g)$ taking together prove the theorem.

Taking $\rho_{p,q}(r) = \rho(p,q)$ $\forall r > r_0$ and $\phi(x) = x^{i/(\rho,q)-A}$, we have the following corollary which gives a formula for lower (p,q)-type t(p,q) in terms of coefficients of an entire function f(z)

Corollary 1. Let $f(z) = \sum_{n=0}^{\infty} C_n W_n(Z)$ Then f(z) be an entire function having (p,q)-order $\rho(p,q)$ $(b < \rho(p,q) < \infty)$ and lower (p,q) type t(p,q) $(0 < t(p,q) < \infty)$ if and only if

$$\frac{t(p,q)}{\gamma(p,q)} = \max_{\{m_k\}}\left\{\liminf_{k\to\infty}\frac{\log^{[p-2]} m_{k-1}}{(\log^{[q-1]}|C_{m_k}|^{-1/m_k})^{\rho(p,q)-A}}\right\}.$$

Finally, we study the subsequence $\{n_k\}$ of n such that for $f(z) = \sum_{n=0}^{\infty} C_n W_n(Z)$, it satisfies

$$\left|C_{n_{k-1}}\right| > \left|C_{n_k}\right| \text{ and } C_n = C_{n_{k-1}} \text{ for } n_{k-1} \le n < n_k. \tag{1.14}$$

The next theorem shows how this sequence influences the growth of $f(z) = \sum_{n=0}^{\infty} C_n W_n(Z)$ in reference to its generalized (p,q) type and generalized lower (p,q)-type.

Theorem 4. Let $f(z)=\sum_{n=0}^{\infty} C_n W_n(Z)$ be an entire function having (p,q)order $\rho(p,q)$ $(b<\rho(p,q)<\infty)$, generalized (p,q)-type T* (p,q) and generalized lower (p,q)-type t* (p,q) and $\{n_k\}$ be the sequence defined by (1.14). Then

$$t^*(p,q)\le T^*(p,q)\liminf_{k\to\infty}\left[\frac{\phi(\log^{[p-2]} n_{k-1})}{\phi(\log^{[p-2]} n_k)}\right]^{\rho(p,q)}, \quad p\ge 3$$

Further, if $\{m_k, j\}$ be the sequence of principal indices satisfying $m_{k-1}\simeq m_k$ as $k\to\infty$ then

$$t^*(2,q)\le T^*(2,q)\liminf_{k\to\infty}\left[\frac{\phi(n_{k-1})}{\phi(n_k)}\right]^{\rho(2,q)-A}.$$

Proof. Let us define a function u(z) such that

$$u(z)=\sum_{n=1}^{\infty}(C_{n-1}-C_n)z^n$$

$$=\sum_{k=1}^{\infty}\alpha_k z^{n_k}$$

where

$$\alpha_k = C_{n_{k-1}} - C_{n_k}.$$

Since the function $g(z)=\sum_{n=0}^{\infty}|C_n|z^n$. has same (p,q)-order as that of f(z), it follows that u(z) has also the same (p,q)-order. By Theorem 1, the generalized (p,q)-type and generalized lower (p,q)-type of u(z) are given by

$$T^*(p,q,f)=T^*(p,q,u) \quad \text{and} \quad t^*(p,q,f)=t^*(p,q,u)$$

Thus, using Theorem by Kasana [3] we have

$$\frac{T^*(p,q,f)}{\gamma(p,q)}=\limsup_{k\to\infty}\left[\frac{\phi(\log^{(p-2)} n_k)}{\log^{(q-1)}\alpha_{n_k}^{-1/n_k}}\right]$$

Considering above formula and Theorem 3, we observe that

$$t^*(p,q)=\gamma(p,q)\max_{\{k_m\}}\left\{\liminf_{m\to\infty}\left[\frac{\phi(\log^{[p-2]} n_{k_{m-1}})}{\log^{[q-1]}\alpha_{n_{k_m}}^{-1/n_{k_m}}}\right]^{\rho(p,q)}\right\}$$

$$\leq \gamma(p,q)\max_{\{k_m\}}\left\{\limsup_{m\to\infty}\left[\frac{\phi(\log^{[p-2]} n_k)}{\log^{[q-1]}\alpha_{n_{k_m}}^{-1/n_{k_m}}}\right]^{\rho(p,q)}\right\}\times$$

$$\max_{\{k_m\}}\left\{\liminf_{m\to\infty}\left[\frac{\phi(\log^{[p-2]} n_{k_{m-1}})}{\phi(\log^{[p-2]} n_{k_m})}\right]^{\rho(p,q)}\right\}$$

$$\leq T^*(p,q)\liminf_{m\to\infty}\left[\frac{\phi(\log^{[p-2]} n_{k-1})}{\phi(\log^{[p-2]} n_k)}\right]^{\rho(p,q)}$$

Similarly, for the case $p = 2$ and $q = 1$ or $q = 2$, let $\{m_k\}$ be the sequence of principal indices such that $m_{k-1} \simeq m_k$ as $k \to \infty$, we have

$$t^*(2,q) \leq T^*(2,q)\liminf_{k\to\infty}\left[\frac{\phi(n_{k-1})}{\phi(n_k)}\right]^{\rho(2,q)-A}.$$

REFERENCES

[1] Juneja, O.P. Kapoor, G.P. and Bajpai, S.K., On the (p,q)-order and lower (p,q)-order of an entire function. J. Reine Angew. Math. 282(1976), 53-67.

[2] Juneja, O.P. Kapoor, G.P. and Bajpai, S.K., On the (p,q)-type and lower (p,q)-type of an entire function. J.Reine Angew. Math. 290(1977), 180-190.

[3] Kasana, H.S., The generalized type of entire functions with index-pair (p,q),Comment. Math. 29 (1990), no-2, 101-108.

[4] Kasana, H.S., Kumar, D. And Srivastava, G.S., On the generalized lower type of entire gap power series U.U.D.M. Report, 16 (1990), 1-2.

[5] Rizvi, S.R.H., Interpolation series and approximation of entire function. The Math, Student 50(1982)-50-57.

[6] Smirnov, V.I. and Lebedev, N.A., Functions of Complex Variables. I liffe books, London, 1968.

[7] Walsh, J.L., Interpolation and Approximation by Rational Function in the Complex Domain, Amer. Math. Soc. Colloq. Publ. Vol. 22,1965.

[8] Winiarski, T.N., Approximation and interpolation of entire functions, Annales Polinici Math. 23 (1970), 259-273.