THE EFFICIENCY ON 2-GENERATORS OF SEMI-DIRECT PRODUCT OF GROUPS

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ABSTRACT

Let G be a semi-direct product of K by A where K and A are both cyclic groups of order n ($n \in N$) and p (p is a prime), respectively. Then we prove that G has an efficient presentation on the minimal number, that is 2, of generators. After all, as an application of our main result, we give the efficiency of the dihedral group D_m and the metacylic group of order 2m ($m \geq 4$ and m is even).

1. INTRODUCTION

1.1. EFFICIENCY

Let G be a finitely presented group, and let $P = \langle x ; r \rangle$ be a finite presentation for G. Then the Euler characteristic of P is defined by $\chi(P) = 1 - |x| + |r|$ where $|\ |$ denotes the number of elements in the set. Let

$$\delta(G) = 1 - rk_z (H_1(G)) + d(H_2(G)), \qquad (1)$$

where $rk_z()$ denotes the Z-rank of the torsion-free part and d( ) means the minimal number of generators. Then, by [3,] [7], [11] for the presentation P, it is always true that $\chi(P) \geq \delta(G)$. We then define

$$\chi(G) = \min\{\chi(P) : P \text{ is a finite presentation for } G\}$$

We then have the following definition.

Definition 1.1. Let G be a group.
i)   A presentation P_0 for G is called minimal if

$$\chi(P_0) \le \chi(P),$$
for all presentations P of G.

ii) A presentation P_0 is called efficient if

$$\chi(P_0) = \delta(G).$$

iii) G is called efficient if
$$\chi(G) = \delta(G).$$

We note that if $\chi(G) \le 0$ then G must be infinite, and if G is finite cyclic then $\chi(G) = 1$.

Examples of efficient groups are finitely generated abelian groups (Epstein [11]), fundamental groups of closed 3-manifolds [11]; also finite groups with balanced presentations (such finite groups have trivial Schur multiplier [12]). Finite metacyclic groups are efficient. This was shown by Beyl [5] and Wamsley [24]. Infinite metacyclic groups however need not be efficient, a result due to Baik and Pride [3] (see also [1]). In [12] Harlander proved that a finitely presented group embeds into an efficient group. For more references on the efficiency see Baik, Pride [2] Beyl, Rosenberger [6], Champbell, Robertson, Williams [8] (and [9]), Çevik [10] Harlander [13], Johnson, Robertson [15], Kenne [17], Kovacs 18 , Swan [23]. Wiegold [26].

There is interest not just in finding efficient presentations, but finding presentations which are efficient on the minimal number of generators (see [25]). So here we examine the efficiency of the semi-direct product and then we show that this semi-direct product has an efficient presentation on 2-generators under some certain conditions (see Theorem 1.4 below). Then we give some consequences of this result on the dihedral groups and metacyclic groups.

There is also interest in finding inefficient groups. The following remark is defined to show that a group is inefficient.

Remark 1.2. Let P be a presentation for the group G, and let $\chi(p) \ne \delta(G)$. Then, by Definition 1.1-(ii), P is an inefficient presentation. However we can still show that P is a minimal presentation for the group G (see [19]). Hence there cannot be an efficient presentation for G. This implies that G is an inefficient group.

An important remark for our result is the fact that the Schur multiplier M(G) of a group G is isomorphic to second integral homology group $H_2(G)$ of that group.

We should also note that the notation $\mathbf{Z}_k$ denotes the cyclic groups of order $k (k \in \mathbf{N})$ in this paper.

1.2. THE DEFINITION OF SEMI-DIRECT PRODUCT

Let A,K be groups, and let θ be a homomorphism defined by
$$\theta : A \to Aut(K), \quad a \to \theta_a$$
for all $a \in A.$. Then the semi-direct product $G = K \rtimes_\theta A$ of K by A is defined as follows.

The elements of G are all ordered pairs (a, k) $(a \in A, \in K)$ and the multiplication is given by
$$(a, k) (a', k') = (aa', (k\theta_{a'})k').$$

Similar definitions of a semi-direct product can be found in [4] or [22]. The proof of the following lemma can be found in [14, Proposition 10.1, Corollary 10.1].

Lemma 1.3. Suppose that $P_K = \langle y; s \rangle$ and $P_A = \langle x; r \rangle$ are presentations for the groups K and A respectively under the maps
$$y \mapsto k_y \quad (y \in y), \quad x \mapsto a_x (x \in X).$$
Then we have a presentation for $G = K \rtimes_\theta A$
$$P = \langle \mathbf{y}, \mathbf{x}; \mathbf{s}, \mathbf{r}, \mathbf{t} \rangle$$
where $t = \{yx\lambda_{yx}^{-1}x^{-1} | y \in y, x \in x\}$ and λ_{yx} is a word on y representing the element $(k_y)\theta_{a_x}$ of K $(a \in A, k \in K, x \in x, y \in y)$.

1.2. THE MAIN THEOREM

Let K be a cyclic group of order $(n \in \mathbf{N})$ with a presentation $P_K = \langle y; y^n \rangle$, and let A be a cyclic group of order p (p is a prime) with a presentation $P_A = \langle x, x^p \rangle$. Then, by Lemma 1.3 a presentation for $G = K \rtimes_\theta A$ is given by
$$P = \langle y, x; y^n = 1, x^p = 1, x^{-1}yx = y^r \rangle, \tag{2}$$
where
  (i) $(r, n) = 1$,
  (ii) $(r-1, nt) = t$ with $t = (r-1, n)$,
  (iii) $t = p$,

(iv) $r^P \equiv 1 \pmod{nt}$,

(v) $|G| = np$,

for $r, t \in \mathbf{N}$.

Thus we have the following main theorem.

Theorem 1.4. The presentation P, as given in (2), is efficient on 2-generators for the group G.

2. PRELIMINARIES AND PROOF OF THE MAIN THEOREM

Though the rest of paper we will assume that K is a cyclic group of order n with a presentation P_K, A is a cyclic group of order p (p a prime) with a presentation P_A and $G = K \rtimes_\theta A$ with a presentation P as given in (2) which the conditions (i), (ii), (iii), (iv) and (v) hold.

In 1904, Schur proved that when B is a finite group then $H_2(B)$ is a finite group whose elements have order dividing the order of B, and when B is a (finite or infinite) cyclic group then $H_2(B) = 1$ (The details of the proof of these facts can be found in [16]). By using this Schur's results, Park (see [21]) proved the following theorem.

Theorem 2.1. Let $G = K \rtimes_\theta A$. Then $H_2(G)$ is a cyclic group of order p.

We know that both A and K are finite abelian groups, and by conditions (ii)-(iii), since $(r-1, n) = p$ then $(n, p) \neq 1$. Therefore it is easy to prove the following lemma (see [10] for the details).

Lemma 2.2. $d(A \oplus K) = d(A) + d(K)$.

Now we can prove our main theorem.

In the first part of the proof we will calculate $\delta(G)$ as given in (1). By condition (v), since G is a finite group then $rk_z(H_1(G)) = 0$, so we will just calculate

$\delta(G) = 1 + d(H_2)(G))$.

But, by Theorem 2.1, since $H_2(G)$ is a cyclic group of order p then it is easy to see that $d(H(G)) = 1$. We then get $\delta(G) = 2$.

In the second part of the proof we need to calculate the Euler characteristic of P. In fact, since $\chi(P) = 1 - (1+1) + (1+1+1) = 2$ in P, then we get

$$\delta(G) = \chi(P),$$

so P is an efficient presentation for G.

In the final part of the proof let us show the efficiency of P is on 2-generators:

If $\theta = 1$ (that is, the trivial homomorphism), then G is the direct product of the groups A and K and so, by Lemma 2.2, d(G)=2. If $\theta \neq 1$,, then G cannot be abelian and hence $d(G) \neq 1$ which gives that d(G)=2.

This completes the proof.

Remark 2.3. Suppose that t=1 in the conditions (ii), (iii) and (iv). Then again in [21] Park proved that $H_2(G)$ is trivial. Thus we get $d(H_2(G)) = 0$. So the above proof implies that P cannot be an efficient presentation for the group G. But if we show that P is a minimal presentation then, by Remark 1.2, we can say that G is an inefficient group.

3. APPLICATIONS OF THE MAIN THEOREM

Example 3.1. Let us take the dihedral group

$$D_{10} = \left\langle y, x; y^{10} = 1, x^2 = 1, x^{-1}yx = y^{-1} \right\rangle$$

of order 20. Then, by [16] $H_2(D_{10}) = \mathbf{Z}_2$ which gives the same result as in Theorem 2.1. Moreover it is easy to see that the five conditions in Section 1.3 hold. Hence, by Theorem 1.4, we say that the above presentation of the group D_{10} is efficient on 2-generators.

Therefore, as an application of Theorem 1.4, we can generalize this example as follows.

Corollary 3.2. The dihedral group

$$D_m = \left\langle y, x; y^m = 1, x^2 = 1, x^{-1}yx = y^{-1} \right\rangle$$

of order 2m ($m \geq 4$ and is even) is efficient.

Proof. It is easy to see that the conditions (i), (ii), (iii), (iv) and (v) in Section 1.3 hold. Also, as a consequence of Theorem 2.1, in [21] Park proved that $H_2(D_m)$ is a cyclic group of order 2. Then, by Theorem 1.4, D_m is efficient on 2-generators.

We should note that for $m \geq 3$ and m is odd, the above proof cannot cover the efficiency of D_m

Remark 3.3. In Example 3.1, one can show that the dihedral group D_{10} is isomorphic to the metacyclic group, say B, of order 20. Thus, by Theorem 1.4, we say that B has an efficient presentation on 2-generators. In fact, we can extend the isomorphism between these two groups to their general form under some conditions.

Therefore, as a consequence of the Theorem 1.4, we have the following well-known result (see also [5] or [24]).

Corollary 3.4. Let B be a metacylic group of order 2m ($m \geq 4$ and m is even) with a presentation

$$P_B = \left\langle a, b; a^m = 1, b^2 = 1, b^{-1}ab = a^{-1} \right\rangle.$$

Then P_B is an efficient presentation on 2-generators for the group B.

Proof. A similar proof can be applied as the proof of Corollary 3.2.

REFERENCES

[1] Y.G. Baik, Generators of the second homotopy module of group presentations with applications, Ph.D Thesis, (1992), University of Glasgow.

[2] Y.G. Baik and S.J. Pride, On the efficiency of coxeter groups, Bull. Of the London Math.Soc. (1997), 29, 32-36.

[3] Y.G. Baik and S.J. Pride, Generators of second homotopy module of presentations arising from group constructions, Preprint, University of Glasgow, 1993.

[4] G. Baumslag, Topics in Combinatorial Group Theory, Lectures in Mathematics, 1993 Birkhauser Verlag.

[5] F.R. Beyl, The Schur-multiplicator of metacyclic groups, Proc. Amer. Math. Soc. 40 (1973),413-418.

[6] F.R. Beyl and G. Rosenberger Efficient Presentations of $GL(2, \mathbf{Z})$ and $PGL(2, \mathbf{Z})$ Groups-St Andrews 1985 (E.F. Robertson and C.M. Campbell eds, LMS Lecture Notes 121, CUP 1986, 135-137.

[7] F.R. Beyl and J.Tappe, Group Extensions, Representations and the Schur Multiplicator. Lecture Notes in Mathematics 958, Springer-Verlag 1982.

[8] C.M. Campbell, E.F. Robertson and P.D. Williams, Efficient Presentation for Finite Simple Groups and Related Groups, Groups-Korea, 1988 (A.C. Kim and B.H. Neumann eds), Lecture Notes in Mathematics 1398, Springer-Verlag, 1989, 65-72.

[9] C.M. Campbell, E.F. Robertson and P.D. Williams, On the Efficieny of Some Direct Powers of Groups, Groups-Canberra 1989 (L.G. Kovacs ed), Lecture Notes in Mathematics 1456, Springer-Verlag 1990, 106-113.

[10] A.Sinan Çevik, Minimality of group and monoid presentations, Ph.D Thesis, (1997), University of Glasgow.

[11] D.B.A. Epstein, Finite presentations of groups and 3-manifolds, Quart. J. Math. Oxford Ser (2), 12, 205-212 (1961).

[12] J. Harlander, Embeddings into efficient groups, Proceedins of the Edinburgh Math. Soc. (1997), 40,313-324.

[13] J. Harlander, Closing the relation gap by direct-product stabilization, J. Alg. 1996, Vol 182, No 2, 511-521.

[14] D.L. Johnson, Presentation of Groups, LMSST Series 15, Cambridge University Press. 1990.

[15] D.L. Johnson and E.F. Robertson, Finite Groups of Deficiency Zero, in Homological Group Theory (C.T.C. Wall ed), LMS Lecture Notes Series 36, CUP 1979, 275-289.

[16] G. Karpilovsky, The Schur Multiplier, LMS Monograms New Series 2, Oxford Science Publications, 1987.

[17] P.E. Kenne, Some new efficient soluble groups, Comm. Alg. 18 (1990), 2747-2753.

[18] L.G. Kovacs, Finite Groups with Trivial Multiplicator and Large Deficiency. Proceedings Groups-Korea 1994 (A.C.Kim and D.L. Johnson (eds)), Walter de Gruyter, 1995, 277-284.

[19] M. Lustig, Fox Ideals, N-Torsion and Applications to Groups and 3-Manifolds, in Two-Dimensional Homotopy and Combinatorial Group Theory (C.Hog-Angeloni, W. Metzler and A.J. Sieradski, editors), CUP, 219-250 (1993).

[20] W. Magnus, A.Karrass, D. Solitar, Combinatorial Group Theory, Second Edition (1966), Dover Publication, New York.

[21] S.A. Park, Projective representations of some finite groups, J. Korean Math. Soc. 22 (1985), 173-180.

[22] J.J. Rotman, Theory of Groups, Wm.C. Brown Publishers, Third Edition, 1988, Iowa.

[23] R.G. Swan, Minimal resolutions for finite groups, Topology 4 (1965), 193-208.

[24] J.W. Wamsley, The deficiency of metacyclic groups, Proc. Amer. Math. Soc. 24 (1970),724-726.

[25] J.W. Wamsley, Minimal presentations for finite groups, Bull. London Math. Soc. 5 (1973), 129-144.

[26] J.Wiegold, The Schur Multiplier: an Elementary Approach, Groups-St Andrews 1981 (ed. By C.M. Campbell and E.F. Robertson, LMS Lecture Note Series 71), 137-154.