Commun. Fac. Sci. Univ. Ank. Series A1
V. 48. pp. 23-31 (1999)

ON THE SPEED OF CONVERGENCE OF THE INTEGRAL OF CAUCHY-STIELTJES TYPE AND GENERALIZED PRIVALOV'S LEMMA*

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(Received May. 22, 1998; Revised May. 21, 1999; Accepted May. 28, 1999)

ABSTRACT

The speed of convergence of the Cauchy-Stieltjes singular integral to its boundary value is established. The main result is the generalization of Tanaka's version of fundamental Privalov's lemma.

1. INTRODUCTION

Let L be a closed rectifiable Jordan curve of length ℓ in the complex z – plane. We denote by $\varphi(s)$ the angle between the positive real axis and the tangent at the point $x(s)$ on L, where s is the arc length parameter. Let $f(s)$ be a complex-valued function of s of bounded variation on the interval $[0,\ell]$. The following integral is called the integral of Cauchy-Stieltjes type

$$F(z)=\frac{1}{2\pi i}\int_L \frac{e^{i\varphi}df(s)}{x-z}=\frac{1}{2\pi i}\int_0^{\ell} \frac{e^{i\varphi(s)}df(s)}{x(s)-z}.$$

If $F(z)\equiv 0$ for z outside L, the integral of Cauchy-Stieltjes type is called Cauchy-Stieltjes integral [3, p. 154].

Let x_0 be the point $x(s_0)$ and L_ε $(\varepsilon>0)$ be the part of L which is left after cutting off the small arc with end points $x(s_0-\varepsilon)$ and $x(s_0+\varepsilon)$. If the limit

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{L_\varepsilon} \frac{e^{i\varphi}df(s)}{x-x_0}$$

exists, then we call it the singular integral at x_0, and we write

* This research was partly supported by the Tübitak, Tbag-1607.

$$\frac{1}{2\pi i}\int_L \frac{e^{i\varphi}df(s)}{x-x_0} = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{L_\varepsilon} \frac{e^{i\varphi}df(s)}{x-x_0}.$$

Choosing a point z on the straight line $z\,x_0$ inclined by angle ψ_0 to the normal at the point x_0, at a distance ε from x_0, i.e., $z = x_0 \pm i\varepsilon e^{i(\varphi_0+\psi_0)}$, consider the difference:

$$F(\varepsilon, x_0, \psi_0) = \frac{1}{2\pi i}\left[\int_L \frac{e^{i\varphi}df(s)}{x-z} - \int_{L_\varepsilon} \frac{e^{i\varphi}df(s)}{x-x_0}\right]$$

which is well defined at the point x_0 on L where a defined tangent exists.

The following fundamental lemma was proved by I.I. Privalov [3]:

Lemma 1 (Privalov). If $f'(s_0)$ exists then the difference $F(\varepsilon, x_0, \psi_0)$ tends to $\frac{1}{2}f'(s_0)$ (or $-\frac{1}{2}f'(s_0)$) uniformly with respect to ψ_0, $|\psi_0| \le \frac{\pi}{2}\theta$, $(0<\theta<1)$, when z tends to x_0 from the inside (from the outside) L, respectively.

This lemma has an important role in the theory of boundary values of analytic functions. The boundary value of the integral of Cauchy-Stieltjes type is obtained with the help of this lemma. Privalov also proved that the difference between the values of the integral of Cauchy-Stieljes type inside and outside L tends to $F'(s_0)$ for all points on the curve L when $z \to L$ (from the inside and outside). Ibragimov and Gadziev [2] established the order of this convergence. In [4], we investigate the speed of convergence in Privalov's lemma.

Tanaka [5] generalized this lemma to the case where L has a corner at x_0, and both $F'_\pm(s_0)$ exist:

Lemma 2 (Tanaka). Let

$$\varphi_1 = \lim_{h\to 0^+}\varphi(s_0-h),\ \varphi_2 = \lim_{h\to 0^+}\varphi(s_0+h),\ \theta = \varphi_2-\varphi_1\ (|\theta|<\pi)$$

and further that f(s) is continuous at $s = s_0$. Put $z = x_0 + \varepsilon e^{i\varphi_1}e^{i\alpha} = x_0 + \varepsilon e^{i\varphi_2}e^{i(\alpha-\theta)}$ for $\varepsilon > 0, |\alpha| < \pi$, where

$$|\cos\alpha| \le q,\ |\cos(\alpha-\theta)| \le q \text{ for a fixed } q\ (0<q<1). \tag{1.1}$$

Then we have

$$F(\varepsilon, x_0, \alpha) = \frac{1}{2\pi i}\left[\int_L \frac{e^{i\varphi}df(s)}{x - z} - \int_{L_\varepsilon} \frac{e^{i\varphi}df(s)}{x - x_0}\right] \to \frac{f'_+(s_0)}{2\pi}[(\theta-\alpha) - \text{sign}(\theta-\alpha)\pi] + \frac{f'_-(s_0)}{2\pi}\alpha$$

as $\varepsilon \to 0$ uniformly with respect to α satisfying (1.1).

In this paper we investigate the speed of convergence in the generalized Privalov's lemma. We obtain the order of convergence, to its boundary value, of the integral of Cauchy-Stieljes type along the Liapunov's curve, having a corner in some point, by using the methods of the theory of singular integrals as in [2]. Our result is a generalization of the Tanaka's lemma.

2. THE ORDER OF CONVERGENCE OF THE INTEGRAL OF CAUCHY-STIELTJES TYPE

In this section we assume that L is a closed Liapunov curve of length ℓ in the complex z-plane, i.e., a curve L on which the angle φ of inclination of its tangent, as a function of the arc length parameter s, satisfies a Hölder condition: $|\varphi(s_1) - \varphi(s_2)| \le c|s_1 - s_2|^\lambda$, where s_1 and s_2 are any points on L, c is a positive constant, and $0 < \lambda < 1$. Notice that a Liapunov curve is always an oriented rectifiable Jordan curve. Assume L has a corner at x_0 and as above f(s) be a complex-valued function of s of bounded variation on the segment $[0, \ell]$. Furthermore, let f(s) be continuous at $s = s_0$ and both $f'_\pm(s_0)$ exist. Following [1], we introduce the function

$$\gamma(s) = \begin{cases} \dfrac{f(s) - f(s_0)}{s - s_0} - f'_+(s_0) & , \quad s > s_0 \\ 0 & , \quad s = s_0 \\ \dfrac{f(s) - f(s_0)}{s - s_0} - f'_-(s_0) & , \quad s > s_0 \end{cases}$$

It is clear that $\lim\limits_{s \to s_0} \gamma(s) = 0$ on $[0, \ell]$. By the definition of $\gamma(s)$,

$$f(s) = f(s_0) + f'_-(s_0)(s - s_0) + \gamma(s)(s - s_0) \quad \text{for} \quad s < s_0,$$

$$f(s) = f(s_0) + f'_+(s_0)(s - s_0) + \gamma(s)(s - s_0) \quad \text{for} \quad s > s_0,$$

and hence

$$df(s) = f'_-(s_0)ds + d[\gamma(s)(s - s_0)] \quad \text{for} \quad s < s_0,$$
$$df(s) = f'_+(s_0)ds + d[\gamma(s)(s - s_0)] \quad \text{for} \quad s > s_0,$$

Then

$$F(\varepsilon, x_0, \alpha) = \frac{1}{2\pi i}\left[\int_L \frac{e^{i\varphi}df(s)}{x - z} - \int_{L_\varepsilon} \frac{e^{i\varphi}df(s)}{x - x_0}\right]$$
$$= \frac{1}{2\pi i}\left[\int_{L_h} \frac{e^{i\varphi}df(s)}{x - z} + \int_{s_0-h}^{s_0+h} \frac{e^{i\varphi}df(s)}{x - z} - \int_{L_\varepsilon} \frac{e^{i\varphi}df(s)}{x - x_0}\right]$$
$$= \frac{1}{2\pi i}\left[\int_{L_h} \frac{e^{i\varphi}df(s)}{x - z} + \int_{s_0-h}^{s_0-\varepsilon} + \int_{s_0-\varepsilon}^{s_0+\varepsilon} + \int_{s_0+\varepsilon}^{s_0+h} \frac{e^{i\varphi}df(s)}{x - z} - \int_{s_0-h}^{s_0-\varepsilon} - \int_{s_0+\varepsilon}^{s_0+h} \frac{e^{i\varphi}df(s)}{x - x_0} - \int_{L_h} \frac{e^{i\varphi}df(s)}{x - x_0}\right].$$

Hence by (2.1),

$$F(\varepsilon, x_0, \alpha) = \frac{1}{2\pi i}\int_{s_0-h}^{s_0-\varepsilon} \frac{e^{i\varphi}\{f'_-(s_0)ds + d[\gamma(s)(s - s_0)]\}}{x - z} + \frac{1}{2\pi i}\int_{s_0+\varepsilon}^{s_0+h} \frac{e^{i\varphi}\{f'_+(s_0)ds + d[\gamma(s)(s - s_0)]\}}{x - z} +$$
$$\frac{1}{2\pi i}\int_{s_0-\varepsilon}^{s_0} \frac{e^{i\varphi}\{f'_-(s_0)ds + d[\gamma(s)(s - s_0)]\}}{x - z} - \frac{1}{2\pi i}\int_{s_0}^{s_0+\varepsilon} \frac{e^{i\varphi}\{f'_+(s_0)ds + d[\gamma(s)(s - s_0)]\}}{x - z} +$$
$$\frac{1}{2\pi i}\int_{s_0-h}^{s_0-\varepsilon} \frac{e^{i\varphi}\{f'_-(s_0)ds + d[\gamma(s)(s - s_0)]\}}{x - x_0} + \frac{1}{2\pi i}\int_{s_0+\varepsilon}^{s_0+h} \frac{e^{i\varphi}\{f'_+(s_0)ds + d[\gamma(s)(s - s_0)]\}}{x - x_0} +$$
$$\frac{1}{2\pi i}\int_{L_h} \frac{(z - x_0)e^{i\varphi}df(s)}{(x - z)(x - x_0)}$$
$$= \frac{f'_-(s_0)}{2\pi i}\left(\int_{s_0-h}^{s_0} \frac{e^{i\varphi}ds}{x - z} - \int_{s_0-h}^{s_0-\varepsilon} \frac{e^{i\varphi}ds}{x - x_0}\right) + \frac{f'_+(s_0)}{2\pi i}\left(\int_{s_0}^{s_0+h} \frac{e^{i\varphi}ds}{x - z} - \int_{s_0+\varepsilon}^{s_0+h} \frac{e^{i\varphi}ds}{x - x_0}\right) +$$
$$\frac{1}{2\pi i}\int_{s_0-\varepsilon}^{s_0+\varepsilon} \frac{e^{i\varphi}d[\gamma(s)(s - s_0)]}{x - z} + \frac{1}{2\pi i}\int_{s_0-h}^{s_0-\varepsilon} \frac{(z - x_0)e^{i\varphi}d[\gamma(s)(s - s_0)]}{(x - z)(x - x_0)} +$$
$$\frac{1}{2\pi i}\int_{s_0+\varepsilon}^{s_0+h} \frac{(z - x_0)e^{i\varphi}d[\gamma(s)(s - s_0)]}{(x - z)(x - x_0)} + \frac{1}{2\pi i}\int_{L_h} \frac{(z - x_0)e^{i\varphi}df(s)}{(x - z)(x - x_0)}.$$
$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \tag{2.2}$$

By an elementary computation, it is seen that the limit values as $\varepsilon \to 0$ of first two term on the right-hand side is

$$I_1 \to \frac{f'_-(s_0)}{2\pi}\alpha, \quad \text{and} \quad I_2 \to \frac{f'_+(s_0)}{2\pi}[(\theta-\alpha)-\operatorname{sign}(\theta-\alpha)\pi].$$

Consequently, the following theorem shows that the rate of convergence to zero of the function $\gamma(s)$ has an essential influence on the order of convergence in the generalized Privalov's lemma.

Theorem. Let L be a closed Liapunov curve, having a corner in the point $x = x_0$ and α, θ satisfies (1.1).Let also for some λ, $(0 \le \lambda < 1)$, $|\gamma(s)| = o(|s-s_0|^{\lambda})$ as $s \to s_0$, and for some μ, ν, $(0 \le \mu, \nu < 1)$,

$$\left|I_1 - \frac{f'_-(s_0)}{2\pi}\alpha\right| = o(\varepsilon^{\mu}) \text{ and } \left|I_2 - \frac{f'_+(s_0)}{2\pi}[(\theta-\alpha)] - \operatorname{sign}(\theta-\alpha)\pi\right| = o(\varepsilon^{\nu}) \text{ as } \varepsilon \to 0.$$

Then the following relation holds uniformly with respect to α

$$\left|F(\varepsilon, x_0, \alpha) - \left(\frac{f'_+(s_0)}{2\pi}[(\theta-\alpha)-\operatorname{sign}(\theta-\alpha)\pi] + \frac{f'_-(s_0)}{2\pi}\alpha\right)\right| = o(\varepsilon^{\beta}) \text{ as } \varepsilon \to 0,$$

where $\beta = \min(\lambda, \mu, \nu)$.

Proof. We define the characteristic function $\chi(x)$ by

$$\chi(x) = \begin{cases} 0, & x \in (x(s_0-\varepsilon), x(s_0+\varepsilon)) \\ 1, & x \in L_{\varepsilon} \end{cases}.$$

Then we have

$$F(\varepsilon, x_0, \alpha) = \frac{1}{2\pi i}\left[\int_L \frac{e^{i\varphi} df(s)}{x-z} - \int_L \frac{e^{i\varphi}\chi(x) df(s)}{x-x_0}\right]$$

$$= \frac{1}{2\pi i}\int_L \left[\frac{e^{i\varphi}}{x-z} - \frac{e^{i\varphi}\chi(x)}{x-x_0}\right] df(s)$$

By the parametric equation of L, we have

$$x - x_0 = x(s) - x(s_0) = (s-s_0)x'_-(s_0) + o(s-s_0) = (s-s_0)(e^{i\varphi_1} + k_1(s)) \text{ for } s < s_0$$

$$x - x_0 = x(s) - x(s_0) = (s-s_0)x'_+(s_0) + o(s-s_0) = (s-s_0)(e^{i\varphi_2} + k_2(s)) \text{ for } s > s_0$$

where $k_1(s) \to 0$ as $s \to s_0^-$, $k_2(s) \to 0$ as $s \to s_0^+$. Let $m_1 = k_1(s)e^{-i\varphi_1}$, $m_2 = k_2(s)e^{-i\varphi_2}$, then we can choose a sufficiently small h such that

$$|m_1|, |m_2| < \frac{1}{2}\sqrt{1-q} \text{ for } |s-s_0| \le h.$$

Therefore

$$|x-x_0| = \left|(s-s_0)e^{i\varphi_1}(1+m_1)\right| > |s-s_0|\,\big|1-|m_1|\big| > |s-s_0|\left(1-\frac{1}{2}\sqrt{1-q}\right) \text{ for } s<s_0$$

$$|x-x_0| = \left|(s-s_0)e^{i\varphi_2}(1+m_2)\right| > |s-s_0|\,\big|1-|m_2|\big| > |s-s_0|\left(1-\frac{1}{2}\sqrt{1-q}\right) \text{ for } s>s_0.$$

Furthermore for $s<s_0$,

$$x-z = x(s)-x(s_0)-\varepsilon e^{i\varphi_1}e^{i\alpha} = (s-s_0)(e^{i\varphi_1}+k_1(s))-\varepsilon e^{i\varphi_1}e^{i\alpha}$$

and then

$$\begin{aligned} |x-z| = \left|e^{i\varphi_1}\left[(s-s_0)(1+m_1)-\varepsilon e^{i\alpha}\right]\right| &\geq \left|(s-s_0)(1+m_1)-\varepsilon e^{i\alpha}\right| \\ &\geq \left|(s-s_0)-\varepsilon e^{i\alpha}\right| - |s-s_0|\,|m_1|. \end{aligned}$$

Since $|\cos\alpha| \leq q < 1$ and $(s-s_0)^2+\varepsilon^2 \geq 2\varepsilon|(s-s_0)|$, we have

$$\left|(s-s_0)-\varepsilon e^{i\alpha}\right| \geq \sqrt{(s-s_0)^2+\varepsilon^2-2|s-s_0|\varepsilon q} \geq \sqrt{(s-s_0)^2+\varepsilon^2}\sqrt{1-q}.$$

Hence

$$|x-z| > \sqrt{1-q}\left(\sqrt{(s-s_0)^2+\varepsilon^2}-\frac{1}{2}|s-s_0|\right) > \sqrt{1-q}\,\max\left(\frac{\varepsilon}{2},\frac{|s-s_0|}{2}\right).$$

Similarly, for $s>s_0$ we obtain

$$|x-z| = \left|e^{i\varphi_2}\left[(s-s_0)(1+m_2)-\varepsilon e^{i(\alpha-\theta)}\right]\right| > \sqrt{1-q}\,\max\left(\frac{\varepsilon}{2},\frac{|s-s_0|}{2}\right).$$

Now let us

$$P(\varepsilon,x_0,\alpha) = \frac{e^{i\varphi}}{2\pi i}\left[\frac{1}{x-z}-\frac{\chi(x)}{x-x_0}\right].$$

By the definition of $\chi(x)$, we have

$$P(\varepsilon,x_0,\alpha) = \begin{cases} \dfrac{e^{i\varphi}}{2\pi i(x-z)}, & |s-s_0| < \varepsilon < h \\[2ex] \dfrac{e^{i\varphi}(z-x_0)}{2\pi i(x-z)(x-x_0)}, & \varepsilon \leq |s-s_0| < h \end{cases}$$

Then $P(\varepsilon,x_0,\alpha)$ has a majorant (see also [2])

$$K(\varepsilon, x_0, \alpha) = \begin{cases} \dfrac{1}{\pi\varepsilon\sqrt{1-q}}, & |s-s_0| < \varepsilon < h \\ \dfrac{2\varepsilon}{\pi\sqrt{1-q}\left(2-\sqrt{1-q}\right)|s-s_0|^2}, & \varepsilon \le |s-s_0| < h \\ \dfrac{\varepsilon}{2\pi \min\limits_{|s-s_0|\ge h}\left(|x-z|\,|x-x_0|\right)}, & |s-s_0| \ge h \end{cases}$$

Using this we can complete the proof by estimating the last four integral in (2.2). Writing

$$\begin{aligned} I_3 &= \frac{1}{2\pi i}\int_{s_0-\varepsilon}^{s_0+\varepsilon} \frac{e^{i\varphi} d[\gamma(s)(s-s_0)]}{x-z} \\ &= \frac{1}{2\pi i}\int_{s_0}^{s_0+\varepsilon} \frac{e^{i\varphi} d[\gamma(s)(s-s_0)]}{x-z} + \frac{1}{2\pi i}\int_{s_0-\varepsilon}^{s_0} \frac{e^{i\varphi} d[\gamma(s)(s-s_0)]}{x-z} \\ &= I_3^{(1)} + I_3^{(2)}. \end{aligned}$$

We see that

$$\begin{aligned} \left|I_3^{(1)}\right| = \left|\int_{s_0}^{s_0+\varepsilon} \frac{e^{i\varphi} d[\gamma(s)(s-s_0)]}{x-z}\right| &\le \frac{1}{\pi\varepsilon\sqrt{1-q}}\int_{s_0}^{s_0+\varepsilon} d[\gamma(s)(s-s_0)] \\ &= \frac{1}{\pi\varepsilon\sqrt{1-q}}\gamma(s)(s-s_0)\Big|_{s_0-\varepsilon}^{s_0}. \end{aligned}$$

Since $|\gamma(s)| \le o(|s-s_0|^\lambda)$ as $s \to s_0$, for every $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ such that $|\gamma(s)| \le \varepsilon|s-s_0|^\lambda = \varepsilon(s-s_0)^\lambda$ whenever $|s-s_0| < \delta$. Hence,

$$\left|I_3^{(1)}\right| \le \frac{1}{\pi\varepsilon\sqrt{1-q}}\varepsilon^{\lambda+2} = \frac{1}{\pi\sqrt{1-q}}\varepsilon^{\lambda+1} = o(\varepsilon^\lambda).$$

By similar argument, it can be easily shown that $\left|I_3^{(2)}\right| = o(\varepsilon^\lambda)$.

Now we choose the sufficiently small number h such that $h \le \delta$. Then

$$I_4 = \frac{1}{2\pi i}\int_{s_0-h}^{s_0-\varepsilon} \frac{e^{i\varphi}(z-x_0) d[\gamma(s)(s-s_0)]}{(x-z)(x-x_0)} = \int_{s_0-h}^{s_0-\varepsilon} P(\varepsilon, x_0, \alpha) d[\gamma(s)(s-s_0)]$$

and therefore

$$|I_4| = \left|\int_{s_0-h}^{s_0-\varepsilon} P(\varepsilon, x_0, \alpha)\, d[\gamma(s)(s-s_0)]\right| \le \int_{s_0-h}^{s_0-\varepsilon} K(\varepsilon, x_0, \alpha)\, d[\gamma(s)(s-s_0)]$$

$$\le \int_{s_0-h}^{s_0-\varepsilon} \frac{2\varepsilon}{\pi\sqrt{1-q}\,(2-\sqrt{1-q})(s-s_0)^2}\, d[\gamma(s)(s-s_0)].$$

By partial integration, we obtain

$$|I_4| \le \left[\frac{2\varepsilon\gamma(s)(s-s_0)}{\pi\sqrt{1-q}\,(2-\sqrt{1-q})(s-s_0)^2}\right]_{s_0-h}^{s_0-\varepsilon} + \frac{4\varepsilon}{\pi\sqrt{1-q}\,(2-\sqrt{1-q})} \int_{s_0-h}^{s_0-\varepsilon} \frac{\gamma(s)(s-s_0)}{(s-s_0)^3}\, ds.$$

Since $|\gamma(s)| \le \varepsilon |s-s_0|^{\lambda} = \varepsilon (s_0 - s)^{\lambda}$ whenever $|s-s_0| < h \le \delta$, we have

$$\left|\int_{s_0-h}^{s_0-\varepsilon} \frac{\gamma(s)(s-s_0)}{(s-s_0)^3}\, ds\right| \le \varepsilon \int_{s_0-h}^{s_0-\varepsilon} \frac{(s_0-s)^{\lambda}}{(s_0-s)^2}\, ds = \varepsilon \int_{s_0-h}^{s_0-\varepsilon} (s_0-s)^{\lambda-2}\, ds = -\varepsilon \frac{(s_0-s)^{\lambda-1}}{\lambda-1}\Bigg|_{s_0-h}^{x_0-\varepsilon}$$

$$= \frac{-\varepsilon^{\lambda} + \varepsilon h^{\lambda-1}}{\lambda - 1}$$

for $(s_0 - h, s_0 - \varepsilon)$. Hence

$$|I_4| \le \frac{2\varepsilon^{\lambda+3}}{\pi\sqrt{1-q}\,(2-\sqrt{1-q})\varepsilon^2} + \frac{2\varepsilon^2 h}{\pi\sqrt{1-q}\,(2-\sqrt{1-q})h^2} + \frac{4\varepsilon(-\varepsilon^{\lambda} + \varepsilon h^{\lambda-1})}{\pi\sqrt{1-q}\,(2-\sqrt{1-q})(\lambda-1)}$$

and then $|I_4| = o(\varepsilon^{\lambda})$.

Similarly, it can be shown that $|I_5| = o(\varepsilon^{\lambda})$.

Finally, we consider the integral

$$I_6 = \int_{L_h} \frac{(z-x_0)e^{i\varphi} df(s)}{(x-z)(x-x_0)} = \int_{L_h} P(\varepsilon, x_0, \alpha)\, df(s).$$

By partial integration,

$$I_6 = [P(\varepsilon, x_0, \alpha) f(s)]_{L_h} - \int_{L_h} f(s)\, d[P(\varepsilon, x_0, \alpha)].$$

Since $|f(s)| \le M$, for some $M > 0$, and $|P(\varepsilon, x_0, \alpha)| \le \dfrac{\varepsilon}{2\pi \min\limits_{|s-s_0|\ge h} (|x-z||x-x_0|)}$ for $|s-s_0| \ge h$, we have

$$|I_6| \le \frac{M\,\varepsilon}{2\pi \min\limits_{|s-s_0|\ge h} \left(|x-z||x-x_0|\right)} + M \int\limits_{L_h} d\left[P(\varepsilon, x_0, \alpha)\right] \le \frac{M\,\varepsilon}{\pi \min\limits_{|s-s_0|\ge h} \left(|x-z||x-x_0|\right)}$$

Hence, we obtain $|I_6| = o(\varepsilon^{\lambda})$ and the proof of theorem is completed.

Note that our theorem implies as a corollary for $\beta = 0$ the lemma of Tanaka. Moreover, if the curve L has not a corner we can obtain the result of I.I. Ibragimov and A.D. Gadziev [2].

The author thanks to Prof. A.D. Gadziev for his encouragement and valuable advices.

REFERENCES

[1] Gadziev, A.D., On the speed of convergence of a class of singular integrals, Izv. Akad. Nauk Azerbaidzan SSR Ser. Fiz-Tekhn. Mat. Nauk (1963), No. 6,27-31.

[2] Ibragimov, I.I. and Gadziev, A.D., On the order of convergence of singular integrals of Cauchy-Stieljes type, Soviet Math. Dokl. Vol. 14 (1973), No.5, 1286-1290.

[3] Privalov, I.I., Boundary properties of analytic functions, GITTL, Moscow, 1950; German Transl. Berlin Deutscher Verlag der Wissenschaften, (1956).

[4] Şerbetçi, A., On the order of convergence of the integral of Cauchy-Stieltjes type and Privalov's lemma, Gazi Üniversitesi Fen Bilimleri Dergisi, 8(1998).

[5] Tanaka, C., On the integral of Cauchy-Stieltjes type and I.I. Privalov's fundamental lemma I-II, Proc. Japan Acad. Vol. 48 (1972), No. 5, 308-314.