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THE SPECTRUM OF THE NON-SELFADJOINT SYSTEM OF DIFFERENCE OPERATORS OF FIRST ORDER

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ABSTRACT

Let L denote the operator generated in $\ell_2(\mathbf{N},\mathbf{C}^2)$ by the difference expression

$$\ell(y)=\begin{pmatrix} a_{n+1}y_{n+1}^{(2)}+b_n y_n^{(2)}+p_n y_n^{(1)} \\ \\ a_{n-1}y_{n-1}^{(1)}+b_n y_n^{(1)}+q_n y_n^{(2)} \end{pmatrix}, \quad a_0=1,\ n\in\mathbf{N}=\{1,2,...\}$$

and the boundary condition

$$y_0^{(1)}=0$$

where (a_n), (b_n), (p_n) and (q_n) are complex sequences and $a_n \neq 0$, $b_n \neq 0$ for all $n\in\mathbf{N}$. In this paper we investigate the continuous spectrum, the eigenvalues and the spectral singularities of L.

1. INTRODUCTION

The spectrum of the selfadjoint difference operator and infinite Jacobi matrices have been studied by various authors ([1], [2], [7]). But the spectral theory of the non-self- adjoint difference operators has not been treated extensively. Some of the problems of the spectral analysis in this area have been investigated by several authors [6], [10], [12].

The spectrum and the spectral expansion in terms of the principal functions of non-selfadjoint differential operators have been considered by Naimark [13]. He showed the existence of spectral singularities in the continuous spectrum of these operators. The spectral singularities of the quadratic pencil of Schrödinger operators has been investigated in [3], [4]. In [5] and [11] the effect of the spectral singularities in the spectral expansion in terms of the principal functions has been studied.

Let $\ell_2(\mathbf{N},\mathbf{C}^2)$ denote the Hilbert space of complex vector-sequences

$y=\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}$, $n \in \mathbf{N}$, with the inner product

$$(y,u)=\sum_{n=1}^{\infty}\left(y_n^{(1)}\overline{u_n^{(1)}}+y_n^{(2)}\overline{u_n^{(2)}}\right).$$

In the space $\ell_2(\mathbf{N},\mathbf{C}^2)$, we consider the operator L generated by the difference expression

$$\ell(y)=\begin{pmatrix} a_{n+1}y_{n+1}^{(2)}+b_n y_n^{(2)}+p_n y_n^{(1)} \\ \\ a_{n-1}y_{n-1}^{(1)}+b_n y_n^{(1)}+q_n y_n^{(2)} \end{pmatrix}$$

and the boundary condition

$$y_0^{(1)}=0,$$

where $a_0=1$, $(a_n),(b_n),(p_n)$ and (q_n) are complex sequences and $a_n \neq 0$, $b_n \neq 0$ for all n.

Let L_0 denote the operator generated in $\ell_2(\mathbf{N},\mathbf{C}^2)$ by the difference expression

$$\ell_0(y)=\begin{pmatrix} \Delta y_n^{(2)}+p_n y_n^{(1)} \\ -\Delta y_{n-1}^{(1)}+q_n y_n^{(2)} \end{pmatrix}$$

and the boundary condition $y_0^{(1)}=0$ where Δ denotes the forward difference operator defined by $\Delta y_n=y_{n+1}-y_n$. The operator L_0 is called the discrete Dirac operator. L_0 is a particular case of L $(a_n\equiv 1,\ b_n\equiv -1,\ n\in\mathbf{N})$. The spectral analysis of L_0 was investigated in [4].

Related with the operator L we will consider the boundary value problem

$$\begin{aligned} &a_{n+1}y_{n+1}^{(2)}+b_n y_n^{(2)}+p_n y_n^{(1)}=\lambda y_n^{(1)} \\ &a_{n-1}y_{n-1}^{(1)}+b_n y_n^{(1)}+q_n y_n^{(2)}=\lambda y_n^{(2)},\quad a_0=1,\ n\in\mathbf{N}, \end{aligned} \tag{1.1}$$

$$y_0^{(1)} = 0 . \tag{1.2}$$

Throughout of this paper we will assume that

$$\sum_{n=1}^{\infty} n\left(|1-a_n| + |1+b_n| + |p_n| + |q_n| \right) < \infty. \tag{1.3}$$

In this paper, we prove the existence of the Jost solution of the equation (1.1) under the condition (1.3). Moreover using the analytic properties of Jost solution of (1.1) we investigate the continuous spectrum, the eigenvalues and the spectral singularities of L.

2. JOST SOLUTION OF (1.1)

Theorem 2.1. Let the condition (1.3) hold. If $\lambda = 2\sin\frac{z}{2}$, then the system (1.1) has a unique solution which is analytic in the half-plane $\operatorname{Im} z > 0$ and continuous upto the real axis having the representation

$$f_0^{(1)}(z) = \alpha_0^{11} e^{iz/2}\left[1 + \sum_{m=1}^{\infty} K_{0m}^{11} e^{imz}\right] - i\alpha_0^{11} \sum_{m=1}^{\infty} K_{0m}^{12} e^{imz}$$

and for $n = 1,2,\ldots$.

$$f(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} \alpha_n^{11} & \alpha_n^{12} \\ \alpha_n^{21} & \alpha_n^{22} \end{pmatrix} \left\{ E_2 + \sum_{m=1}^{\infty} K_{nm} e^{imz} \right\} \begin{pmatrix} e^{iz/2} \\ -i \end{pmatrix} e^{inz} \tag{2.1}$$

where

$$E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_{nm} = \begin{pmatrix} K_{nm}^{11} & K_{nm}^{12} \\ K_{nm}^{21} & K_{nm}^{22} \end{pmatrix}.$$

Moreover the inequality

$$\left|K_{nm}^{ij}\right| \le C\sigma\left(n + \left[\left|\frac{m}{2}\right|\right]\right), \quad i,j = 1,2, \tag{2.2}$$

holds for all m,n where

$$\sigma(n) = \sum_{k=n}^{\infty} \left(|1-a_k| + |1+b_k| + |p_k| + |q_k| \right), \tag{2.3}$$

and $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$ and C is a constant.

Proof. Substituting the function f(z) given by (2.1) in (1.1) and equalizing the corresponding coefficients of e^{inz} we have,

$$\alpha_n^{12} = 0 \ , \quad n \in \mathbf{N}$$

$$\alpha_n^{11} = \left\{ \prod_{k=n+1}^{\infty} (-1)^{n-k} b_k a_{k-1} \right\}^{-1} , \quad n \in \mathbf{N} \cup \{0\}$$

$$\alpha_n^{22} = \left\{ b_n \prod_{k=n+1}^{\infty} (-1)^{n-k+1} b_k a_{k-1} \right\}^{-1} , \quad n \in \mathbf{N} \tag{2.4}$$

$$\alpha_n^{21} = \alpha_n^{22} \left\{ p_n + \prod_{k=n+1}^{\infty} (p_k + q_k) \right\}, \quad n \in \mathbf{N}$$

and

$$K_{n1}^{12} = -\sum_{k=n+1}^{\infty} (p_k + q_k) \ ,$$

$$K_{n1}^{11} = \sum_{k=n+1}^{\infty} (b_k^2 - 1) + \sum_{k=n+1}^{\infty} a_k (a_{k+1} - 1) + \sum_{k=n+1}^{\infty} (a_k - 1) + \sum_{k=n+1}^{\infty} (p_k + q_k) K_{k1}^{12} - \sum_{k=n+1}^{\infty} p_k q_k \ ,$$

$$K_{n1}^{22} = -1 + a_{n+1} a_n + K_{n1}^{12} K_{n1}^{12} + K_{n1}^{11}$$

$$K_{n1}^{21} = -\sum_{k=n}^{\infty} \left(q_{k+1} + K_{k1}^{12} \right) \left\{ a_{k+1} a_k + q_{k+1} (p_{k+1} + q_{k+1}) + q_{k+1} K_{k1}^{12} - K_{k+1,1}^{11} + b_{k+1}^2 - 1 \right\}$$

$$+ \sum_{k=n+1}^{\infty} q_k K_{k1}^{22} - \sum_{k=n+1}^{\infty} b_k^2 p_k \ ,$$

$$K_{n2}^{12} = a_{n+1} a_n \left(p_{n+1} - K_{n+1,1}^{12} \right) + K_{n1}^{12} K_{n1}^{11} + K_{n1}^{12} - K_{n1}^{21} \ ,$$

$$K_{n2}^{11} = \sum_{k=n+1}^{\infty} a_{k+1}a_k \left\{\left(p_k - K_{k+1,1}^{12}\right)K_{k+1,1}^{12} + K_{k+1,1}^{22}\right\} + \sum_{k=n+1}^{\infty}(b_k^2 - 1)K_{k1}^{11}$$

$$+ \sum_{k=n+1}^{\infty}\left(1 + K_{k1}^{12}\right)K_{k2}^{12} - \sum_{k=n+1}^{\infty}\left(q_k K_{k1}^{11} - K_{k1}^{12}\right)\left(p_k - K_{k1}^{12}\right) - \sum_{k=n+1}^{\infty} q_k K_{k1}^{12}$$

$$K_{n2}^{22} = a_{n+1}a_n\left(p_{n+1} - K_{n+1,1}^{12}\right)K_{n+1,1}^{12} + a_{n+1}a_n K_{n+1,1}^{22} + K_{n1}^{12}K_{n2}^{12} - K_{n1}^{11} + K_{n2}^{11}$$

$$K_{n2}^{21} = \sum_{k=n}^{\infty} a_{k+1}a_k\left\{\left(p_{k+1} - K_{k+1,1}^{12}\right)K_{k+1,1}^{11} + K_{k+1,1}^{21}\right\} + \sum_{k=n}^{\infty}\left\{K_{k1}^{12}K_{k2}^{11} + K_{k2}^{12} - K_{k+1,1}^{21}\right\}$$

$$+ \sum_{k=n+1}^{\infty}\left(q_k K_{k2}^{22} - b_k^2 K_{k2}^{12}\right) + \sum_{k=n+1}^{\infty}\left(q_k K_{k2}^{12} + K_{k2}^{11} - K_{k1}^{11}\right)\left(p_k - K_{k1}^{12}\right)$$

and for $m \geq 3$,

$$K_{nm}^{12} = -\sum_{k=n+1}^{\infty} a_{k+1}a_k\left\{(p_{k+1} - K_{k+1,1}^{12})K_{k+1,m-2}^{11} + K_{k+1,m-2}^{21}\right\}$$

$$- \sum_{k=n+1}^{\infty} K_{k1}^{12}K_{k,m-1}^{11} - \sum_{k=n+1}^{\infty} q_k K_{k,m-1}^{22} + \sum_{k=n+1}^{\infty}(b_k^2 - 1)K_{k,m-1}^{12} \tag{2.5}$$

$$- \sum_{k=n+1}^{\infty}(q_k K_{k,m-1}^{12} - K_{k,m-2}^{11} + K_{k,m-1}^{11})(p_k - K_{k1}^{12})$$

$$K_{nm}^{11} = \sum_{k=n+1}^{\infty} a_{k+1}a_k\left\{(p_{k+1} - K_{k+1,1}^{12})K_{k+1,m-1}^{12} + K_{k+1,m-1}^{22}\right\} + \sum_{k=n+1}^{\infty} K_{k1}^{12}K_{km}^{12}$$

$$+ \sum_{k=n+1}^{\infty}(b_k^2 - 1)K_{k,m-1}^{11} - \sum_{k=n+1}^{\infty}(q_k K_{k,m-1}^{11} + K_{k,m-1}^{12} - K_{km}^{12})(p_k - K_{k1}^{12})$$

$$- \sum_{k=n+1}^{\infty} q_k K_{k,m-1}^{21} - \sum_{k=n+1}^{\infty} K_{k,m-1}^{22}$$

$$K_{nm}^{22} = a_{n+1}a_n(p_{n+1} - K_{n+1,1}^{12})K_{n+1,m-1}^{12} + a_{n+1}a_n K_{n+1,m-1}^{22} + K_{n1}^{12}K_{nm}^{12} - K_{n,m-1}^{11} + K_{nm}^{11}$$

$$K_{nm}^{21} = a_{n+1}a_n(p_{n+1} - K_{n+1,1}^{12})K_{n+1,m-1}^{11} + a_{n+1}a_n K_{n+1,m-1}^{21} + K_{n1}^{12}K_{nm}^{11} + K_{nm}^{12} - K_{n,m+1}^{12}$$

for $n \in \mathbf{N} \cup \{0\}$.

Since (1.3) holds, we uniquely have α_n^{ij} and K_{mn}^{ij} , for $i, j = 1,2$, depending on a_n, b_n, p_n and q_n. Using (2.5) we can also prove, by induction, that K_{mn}^{ij} satisfies the inequality (2.2). So the proof is completed.

3. THE SPECTRUM OF L

Let us define $P_0 = \{z \mid z = \eta + i\tau,\ \tau > 0,\ 0 \le \eta < 4\pi\}$ and $P = P_0 \cup [0,4\pi)$ in complex plane and let $\sigma_c(L)$, $\sigma_d(L)$ and $\sigma_{ss}(L)$ denote the continuous spectrum, discrete spectrum (the set of eigenvalues) and spectral singularities of the operator L, respectively.

Theorem 3.1. If the condition (1.3) holds, then $\sigma_c(L) = [-2,2]$.

Proof. Let L_1 and L_2 denote the operators generated in $\ell_2(\mathbf{N},\mathbf{C}^2)$ by the difference expressions

$$\ell_1(y) = \begin{pmatrix} y_{n+1}^{(2)} - y_n^{(2)} \\ y_{n-1}^{(1)} - y_n^{(1)} \end{pmatrix}$$

and

$$\ell_2(y) = \begin{pmatrix} (1 - a_{n+1})y_{n+1}^{(2)} + (1 + b_n)y_n^{(2)} + p_n y_n^{(1)} \\ (1 - a_{n-1})y_{n-1}^{(1)} + (1 + b_n)y_n^{(1)} + q_n y_n^{(2)} \end{pmatrix}$$

respectively. It is evident that, $L = L_1 + L_2$ and

$$L_1 = L_1^*,\ \sigma(L_1) = \sigma_c(L_1) = [-2,2]$$

where $\sigma(L_1)$ denotes the spectrum of L_1. Under the condition (1.3), L_2 is a compact operator in $\ell_2(\mathbf{N},\mathbf{C}^2)$. So Weyl-von Neumann Theorem ([9], p.13) gives the result.

Theorem 3.2. If (1.3) holds, then

$$\sigma_p(L) = \left\{\lambda \mid \lambda = 2\sin\frac{z}{2},\ z \in P_0,\ \beta(z) = 0\right\} \tag{3.1}$$

$$\sigma_{ss}(L)=\left\{\lambda \mid \lambda=2\sin\frac{z}{2},\ z\in(0,4\pi)\setminus\{2\pi\},\ \beta(z)=0\right\} \tag{3.2}$$

where $\beta(z)=f_0^{(1)}(z)e^{-\frac{iz}{2}}$.

Proof. Using Theorem 2.1, we get that the function $f_0^{(1)}(z)$ satisfies the following asimptotic equality

$$f_0^{(1)}(z)=\alpha_0^{11}e^{\frac{iz}{2}}[1+o(1)],\ z\in P_0,\ |z|\to\infty. \tag{3.3}$$

From (3.3) and the definition of eigenvalues of L we obtain

$$\sigma_p(L)=\left\{\lambda \mid \lambda=2\sin\frac{z}{2},\ z\in P_0,\ f_0^{(1)}(z)=0\right\}.$$

Since the operator L is bounded under the hypothesis $\infty\notin\sigma_p(L)$. Consequently we get

$$\sigma_p(L)=\left\{\lambda \mid \lambda=2\sin\frac{z}{2},\ z\in P_0,\ f_0^{(1)}(z)=0\right\}\setminus\{\infty\}$$

$$=\left\{\lambda \mid \lambda=2\sin\frac{z}{2},\ z\in P_0,\ \beta(z)=0\right\}.$$

Let us consider the set

$$M=\left\{\lambda \mid \lambda=2\sin\frac{z}{2},\ z\in(0,4\pi)\setminus\{2\pi\},\ \beta(z)=0\right\}$$

It is clear that

$$M\not\subset\sigma_p(L),\ M\subset[-2,2]=\sigma_c(L).$$

Therefore, by the definition of spectral singularities we find that

$$M=\sigma_{ss}(L).$$

So the proof is completed.

Theorem 3.3. If the condition (1.3) holds, then $\sigma_d(L)$ is bounded, is no more than countable and its limit points can lie only in [-2,2].

Proof. From (2.1) and (2.2) we find that $\beta(z)$ is analytic in $\operatorname{Im} z > 0$, continuous in $\operatorname{Im} z \geq 0$ and

$$\beta(z) = \alpha_0^{11}[1 + o(1)], \; z \in P_0, \; |z| \to \infty. \tag{3.4}$$

Using (3.1), (3.4) and uniqueness theorem of analytic functions we have the proof ([8]).

Theorem 3.4. Under the condition (1.3), $\sigma_{ss}(L) \subset [-2,2]$, $\sigma_{ss}(L)$ is closed and its linear Lebesgue measure is zero.

Proof. From (3.3), it is clear that $\sigma_{ss}(L) \subset [-2,2]$.

On the other hand, since $\beta(z)$ is continuous on $[0,4\pi)$, $\sigma_{ss}(L)$ is closed.

If the linear Lebesgue measure of $\sigma_{ss}(L)$ could be positive, it had to be that $\beta(z) \equiv 0$, which is analytic in P_0. Since $\beta(z) \not\equiv 0$ on P, it is impossible ([8]).

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REFERENCES

[1] N.I. Akhiezer, "The Classical Moment Problem and Some Related Questions in Analysis", Ungar, New York, 1965.

[2] F.V. Atkinson, "Discrete and Continuous Boundary Problems", Academic Press, New York, 1964.

[3] E. Bairamov, Ö. Çakar and A.O. Çelebi, Quadratic Pencil of Schrödinger Operators with Spectral Singularities Discretre Spectrum and Principal Functions. J. Math. Anal. Appl. 216, 303-320 (1997).

[4] E. Bairamov, Ö. Çakar and A.M. Krall, Spectrum and Spectral Singularities of a Quadratic Pencil of a Scrödinger Operator with a General Boundary Condition, J. Differential Eq. 151 (1999), 252-267.

[5] E. Bairamov, Ö. Çakar and A.M. Krall, An Eigenfunctions Expansion for a Quadratic Pencil of Schrödinger Operator with Spectral Singularities, J. Differential Eq. 151 (1999), 268-289.

[6] E. Bairamov and A.O. Çelebi, Spectrum and Spectral Expansion for the Non-selfadjoint Discrete Dirac Operators, Quart. J. Math. Oxford (2), 50 (1999), 371-384.

[7] Yu.M. Berezanski,"Expansion in Eigenfunctions of Selfadjoint Operators", AMS, Providence, 1968.

[8] E.P. Dolzhenko, Boundary Value Uniqueness Theorems for Analytic Functions, Math. Notes 25, No 6, (1979) 437-442.

[9] I.M. Glazman,"Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators", Jerusalem, 1965.

[10] Yu.L. Kisakevich, On an Inverse Problem for Non-Selfadjoint Difference Operators, Math. Notes, 11, No 6, (1972), 402-407.

[11] V.E. Lyance, A Differential Operator with Spectral Singularities I, II, AMS Translations, (2) 60, (1967), 185-225, 227-283.

[12] F.G. Maksudov, B.P. Allakhverdiev and E. Bairamov, On the Spectral Theory of a Non-Selfadjoint Operator Generated by an Infinite Jacobi Matrix, Soviet Math. Dokl. 43, (1991), 78-82.

[13] M.A. Naimark, Investigation of the Spectrum and the Expansion in Eigenfunctions of a Non-Selfadjoint Operator of the Second Order on a Semi-axis, AMS Translations, (2), 16, (1960), 103-193.