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THE CURVATURE CENTER OF THE CURVES ON A HYPERSURFACE IN LORENTZIAN n-SPACE L^n

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ABSTRACT

In this paper, we define i^{th} e-curvature function, r^{th} $(a_1,...,a_r)$-curvature center and $(n-r)$-curvature hyperplane for the curve of Lorentzian space. We prove that the locus of centers of spheres that has p as the r-multiple contact point with curve is the $(n-r)$-curvature hyperplane in the n-dimensional Lorentzian space. We also consider some special cases. In the final section we calculate the $r^{th}(a_1,...,a_r)$-curvature centers $C_r(t)(r=2$ or $3)$ of some special curves.

KEYWORDS

Hypersurface, Lorentzian space.

AMS SUBJECT CLASSIFICATION : 53C40.

1. INTRODUCTION

Let V be an n-dimensional vector space over $\mathbf{R}$ and let $\langle,\rangle$ be an inner product on V. The subspace dimension with maximal dimension in which the inner product is defined negative is called the index of vector space V [7]. If the kernel of this inner product is $N_v=\{0\}$ the inner product is called nondegenerate [2].

If the index of the inner product that is nondegenerate on V is 1, V is called Lorentzian vector space and the inner product is called Lorentzian inner product [7].

The manifold which is determined with this geometrical structure is called Lorentzian manifold or Semi-Riemannian manifold with index 1, [7].

Let L^n be a Lorentzian space and the inner product $\langle,\rangle$ on it. Then a vector X of L^n is called

i) time-like if $\langle x,x\rangle < 0$,

ii) space-like if $\langle x,x\rangle > 0$, or $x=0$,

iii) null or null vector if $\langle x,x\rangle = 0$, $x \neq 0$, [7].

In [5] the present author consider the curvature center of the curves on a hypersurface M^n in $\mathbf{E}^{n+1}$ which was partially contained some results from [4].

In the present study we consider the curvature center of the curves on a hypersurface in a n-dimensional Lorentzian space L^n. We have shown that the locus of centers of spheres that has $\alpha(s)$ as the r-multiple contact point with α are on the $(n-r)$-curvature hyperplane $D_{(n-r)}(s)$. We also consider some special cases for the $r^{th}(a_1,...,a_r)$-curvature centers $C_r(t)$ of the curve $\alpha : I \to L^n$. In the final section we calculate the $r^{th}(a_1,...,a_r)$ -curvature centers $C_r(t)$ of some special curves.

2. BASIC DEFINITIONS AND PROPERTIES OF $\mathbf{L}^n$

The Frenet curvatures and Frenet equations of L^n can be defined as follows.

Definition 2.1. Let $M \subset L^n$ and $\alpha : I \to L^n$ be a curve on L^n and $k_1^\alpha,...,k_{(n-1)}^\alpha$ the Frenet curvatures of α. Then for the unit tangent vector $V=\alpha'(t)$ over M the i^{th} e-curvature function m_i of M is defined by (see [4])

$$m_i = \left\{\begin{array}{lll} 0 & , & i=1 \\ (\varepsilon_1\varepsilon_2)/k_1^\alpha & , & i=2 \\ [V_1(m_{(i-1)}) + \varepsilon_{(i-2)}k_{(i-2)}^\alpha m_{(i-2)}](\varepsilon_i)/k_{(i-1)}^\alpha & , & 2<i\leq n \end{array}\right\},$$

where ε_i the sign of V_i which is $+1$ if the vector V_i is space-like and -1 if the vector V_i is time-like; in other words, $\varepsilon_i = \langle V_i, V_i\rangle$.

Lemma 2.1. [3] Let B be the second fundamental form of the surface M and U is a open neighborhood of a point p. Then for $p \in U$

i) For any two unit vectors X,Y in the tangent space T_pM

$$\langle B(X,X),B(X,X)\rangle = \langle B(Y,Y),B(Y,Y)\rangle.$$

ii) If X,Y are orthonormal vectors in the tangent space T_pM

$$\langle B(X,X),B(X,Y)\rangle = 0,$$
$$\langle B(X,X),B(X,X)\rangle = \langle B(Y,Y),B(Y,Y)\rangle$$

and

$$\langle B(X,X),B(X,X)\rangle = \langle B(X,X),B(Y,Y)\rangle + 2\langle B(X,Y),B(X,Y)\rangle.$$

Lemma 2.2. [1] Let $f : M_r^n \to \mathbf{R}_s^N$ be a planar geodesics immersion and X, Y be two orthogonal non-null vectors in $T_p M$. Then

$$\langle B(X,X), B(X,Y) \rangle = 0.$$

If X, Y are unit vectors, then $\langle B(X,X), B(Y,Y) \rangle + 2\langle B(X,Y), B(X,Y) \rangle = (-1)^d L$, where $d = \langle X,X \rangle \langle Y,Y \rangle$ and $L = \langle B(X,X), B(X,Y) \rangle =$ constant.

Furthermore, let $f : M_r^n \to \mathbf{R}_s^N$ be an isometric immersion. Then f is said to be a planar geodesics immersion if the image of each geodesic of M lies in a 2-plane of $\mathbf{R}_s^N$.

Theorem 2.3. [4] Let $M \subset L^n$, $\dim M = 2$ and X, Y be two orthonormal vectors in the tangent space $T_p M$ at $p \in M$. If M has planar geodesics and $B(X,X) = B(Y,Y)$, then $B(X,Y) = 0$, where B is the second fundamental form of M.

Definition 2.2. Let $M \subset L^n$ be a 2-surface and U be an open neighborhood of $p \in M$. Then for an r-plane Π which passed through the point $p \in M$ defined by $U \cap \Pi$ regular curve α at the point $p \in M$ is called the section curve determined by Π. If the plane Π is orthogonal to M then the section curve is called the normal section curve and for a tangent vector X_p while $X_p \in \Pi$ the section curve determined by Π is called the section curve passed through the point X_p, [4].

Definition 2.3. Let $\alpha : I \to \mathbf{R}_1^n = L^n$ be a non-null curve. If $m_1, ..., m_n$ denote the e-curvature functions of α and $\{V_1, ..., V_n\}$ the Frenet frame field of α then the point

$$C_r(t) = \left(\alpha + \sum_{j=2}^{r} a_j m_j V_j \right)(t), \quad a_j = \pm 1 \tag{2.1}$$

is called r^{th} $(a_1, ..., a_r)$-curvature center of α at the point $\alpha(t)$, [4]. The $(n-r)$-hyperplane which is spanned by $Sp\{V_{r+1}(t), ..., V_n(t)\}$ and passed through the point $C_r(t)$ is called the $(n-r)$-curvature hyperplane of α at the point $\alpha(t)$ and is denoted by $D_{(n-r)}(t)$, where m_j and V_j are the e-curvature functions and Frenet vectors of α, respectively.

Definition 2.4. Let $M \subset L^n$ be a 2-surface and $\alpha : I \to L^n$. A normal section curve determined by $V_p \in T_p M$ which is non-null then

$$C_r^N(t) = \left(\alpha + \sum_{j=2}^{r} a_j^N m_j^N \xi_{(j-1)} \right)(t), \quad a_j^N \in IR, \quad (2 \le r \le (n-2))$$

is called r^{th} $(a_1^N,...,a_r^N)$-curvature center of normal section curve α determined by V_p of M at p. Here $m_1^N,...,m_n^N$ and $k_1^N,...,k_n^N$ are the e-curvature functions and Frenet curvatures of the non-null normal section curve respectively. Furthermore the Frenet frame field of the curve is $\{V_1^N, \xi_1,...,\xi_{(n-1)}\}, V_j^N = \xi_{(j-1)},\ 2 \le j \le (n-1)$, [4].

Theorem 2.4. [6] If $\beta : I \to \mathbf{R}_1^n$ is non-null curve and the Frenet frame of β is $\{V_1,...,V_s\}$ then

$$V_1' = \nabla_{v1} V_1 = \varepsilon_2 k_1 V_2,$$
$$V_i' = \nabla_{v1} V_i = -\varepsilon_{(i-1)} k_{(i-1)} V_{(i-1)} + \varepsilon_{(i+1)} k_{(i+1)} V_{(i+1)},$$
$$V_s' = -\varepsilon_{(s-1)} k_{(s-1)} V_{(s-1)},$$

where $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1,\ 1 \le i \le s$, and k_i are curvature functions of β.

Theorem 2.5. Let $M \subset L^n$ be a surface whose geodesics α are planar. The locus of centers of spheres that has $\alpha(s)$ as the r-multiple contact point with α are on the $(n-r)$ -curvature hyperplane $D_{(n-r)}(s)$.

Proof. Let us define the function

$$f(\alpha(s)) = <\alpha(s) - c, \alpha(s) - c> - r^2.$$

If $\alpha(s)$ is the r-multiple contact point of the curve α and the sphere $S_r(c)$. Then

$$f(\alpha(s)) = f'(\alpha(s)) = ... = f^r(\alpha(s)) = 0.$$

Therefore

(2.2) $$f'(\alpha(s)) = <\alpha'(s), \alpha(s) - c> = <V_1(s), \alpha(s) - c> = 0$$

and generally for $n = 2,3,...,r$

(2.3) $$f^n(\alpha(s)) = <V_n(s), \alpha(s) - c> = -(m_{n-1}' + \varepsilon_{n-2} k_{n-2}(s) m_{n-2}) \frac{\varepsilon_n}{k_{n-1}(s)} = -m_n.$$

Let C be the curvature center of the sphere $S_r(c)$. Consider the vector

$$\overrightarrow{\alpha(s)C} = C - \alpha(s) = \sum_{j=1}^{n} a_j \lambda_j V_j(s)$$

or similarly

(2.4) $$C = \alpha(s) + \sum_{j=1}^{r} a_j \lambda_j V_j(s) + \sum_{j=r+1}^{n} a_j \lambda_j V_j(s)$$

where $\lambda_j \in \mathbf{R}$ are arbitrary parameters and $\lambda_i = \langle V_i(s),\ c - \alpha(s) \rangle$.

On the other hand form (2.2) we get

$$\lambda_1 = \langle V_1(s),\ \alpha(s) - c \rangle = 0.$$

From (2.3) we also get

$$-\lambda_n = \langle V_n(s), \alpha(s)-c\rangle = -m_n \Rightarrow \lambda_n = m_n, \quad 2 < n \leq r.$$

Thus, (2.4) becomes

$$C = \alpha(s) + \sum_{j=1}^{r} a_j m_j V_j(s) + \sum_{j=r+1}^{n} a_j m_j V_j(s)$$

This completes the proof of the theorem.

3. SOME SPECIAL CASES

Consider r^{th} $(a_1,...,a_r)$-curvature centers $C_r(t)$ of the curve $\alpha : I \to L^n$ defined by (2.1). We consider the following cases:

Case I : i) Let $a_j = +1$ and $r = 2$ then the curvature hyperplane $D_{(n-2)}(s)$ is defined by

$$\{\ (\alpha + m_2 V_2)(s) + \lambda_3 V_3(s) + ... + \lambda_n V_n(s)\ |s \in I,\ \lambda_3,...,\lambda_n \in \mathbf{R}\}$$

which is the locus of the $(n-1)$-sphere centers those have $\alpha(s)$ as an 2-multiple common point with α passes through the point $\alpha(s)$. Thus the $(n-2)$-hyperplane $D_{n-2}(s)$ is an affine subspace that associated with $Sp\{V_3,...,V_n\}$ and passes through $C_2(s)$.

ii) For $r = 3$, $D_{n-3}(s)$ is defined by

$$\{(\alpha + m_2 V_2 + m_3 V_3)(s) + \lambda_4 V_4(s) + ... + \lambda_n V_n(s)\ |s \in I,\ \ \lambda_4,...,\lambda_n \in \mathbf{R}\}$$

which is the locus of the $(n-2)$-sphere centers those have $\alpha(s)$ as an 3-multiple common point with α passes through the point $\alpha(s)$. Thus the $(n-3)$-hyperplane $D_{n-3}(s)$ is an affine subspace that associated with $Sp\{V_4,...,V_n\}$ and passes through $C_3(s)$.

iii) For $r = 4$, $D_{n-4}(s)$ is defined by

$$\{(\alpha + m_2 V_2 + m_3 V_3 + m_4 V_4)(s) + \lambda_5 V_5(s) + ... + \lambda_n V_n(s)\ \ |s \in I,\ \lambda_n \in \mathbf{R}\}$$

which is the locus of the $(n-3)$-sphere centers those have $\alpha(s)$ as an 4-multiple common point with α passes through the point $\alpha(s)$. Thus the $(n-4)$-hyperplane $D_{n-4}(s)$ is an affine subspace that associated with $Sp\{V_5,...,V_n\}$ and passes through $C_4(s)$.

iv) Similarly, for $r = (n-1)$, $D_1(s)$ is defined by

$$\{(\alpha + m_2 V_2 + m_3 V_3 + m_4 V_4 + ... + m_{(n-1)} V_{(n-1)})(s) + \lambda_n V_n(s)\ \ |s \in I,\ \lambda_n \in \mathbf{R}\}$$

which is the locus of the 2-sphere centers those have $\alpha(s)$ as an $(n-1)$-multiple common point with α passes through the point $\alpha(s)$. Thus the 1-hyperplane $D_1(s)$ is an affine subspace that associated with $Sp\{V_n\}$ and passes through $C_{(n-1)}(s)$,

v) For $r = n$, $D_0(s)$ is defined by

$$\left\{(\alpha + m_2V_2 + m_3V_3 + m_4V_4 + \ldots + m_nV_n)(s) \;\middle|\; s \in I\right\}$$

which is the locus of the 1-sphere centers those have $\alpha(s)$ as an n-multiple common point with α passes through the point $\alpha(s)$. Thus the 0-hyperplane $D_0(s)$ that is $C_n(s)$ denoted point.

Case II: i) Let $a_j = -1$ and $r = 2$ then the curvature hyperplane $D_{n-2}(s)$ is defined by

$$\left\{(\alpha - m_2V_2)(s) - \lambda_3V_3(s) - \ldots - \lambda_nV_n(s) \;\middle|\; s \in I,\ \lambda_3,\ldots,\lambda_n \in \mathbf{R}\right\}$$

ii) For $r = 3$, $D_{n-3}(s)$ is defined by

$$\left\{(\alpha - m_2V_2 - m_3V_3)(s) - \lambda_4V_4(s) - \ldots - \lambda_nV_n(s) \;\middle|\; s \in I,\ \lambda_4,\ldots,\lambda_n \in \mathbf{R}\right\}$$

iii) For $r = 4$, $D_{n-4}(s)$ is defined by

$$\left\{(\alpha - m_2V_2 - m_3V_3 - m_4V_4)(s) - \lambda_5V_5(s) - \ldots - \lambda_nV_n(s) \;\middle|\; s \in I,\ \lambda_5,\ldots,\lambda_n \in \mathbf{R}\right\}$$

iv) Similarly, for $r = (n-1)$, $D_1(s)$ is defined by

$$\left\{(\alpha - m_2V_2 - m_3V_3 - m_4V_4 - \ldots - m_{(n-1)}V_{(n-1)})(s) - \lambda_nV_n(s) \;\middle|\; s \in I,\ \lambda_n \in \mathbf{R}\right\}$$

v) For $r = n$, $D_0(s)$ is defined by

$$\left\{(\alpha - m_2V_2 - m_3V_3 - m_4V_4 - \ldots - m_nV_n)(s) \;\middle|\; s \in I\right\}$$

4. EXAMPLES

In this section we will calculate the r^{th} $(a_1,\ldots,a_r)$-curvature centers $C_r(t)$ of some special curves.

Example 4.1. Let

$$\alpha(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}\right).$$

Since

$$V_1(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|} = \alpha'(s),\ V_2(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$$

we get

$$k_1(s) = \langle V_1'(s), V_2(s)\rangle = \langle \alpha''(s), V_2(s)\rangle = \frac{1}{\sqrt{2}},\quad (V_1(s) = \alpha'(s))$$

where $\alpha'(s), \alpha''(s)$ the first and second differential of $\alpha(s)$ respectively. If the vector V_1 is time-like and $a_j = +1$ then,

$$\begin{aligned} C_2(s) &= (\alpha + m_2 V_2)(s) \\ &= \alpha(s) + \left(\frac{\varepsilon_1 \varepsilon_2}{k_1(s)}\right) V_2(s) \\ &= \alpha(s) - \frac{1}{k_1(s)} V_2(s), \quad \varepsilon_i = \langle V_i, V_i \rangle \\ &= 2\alpha(s). \end{aligned}$$

If the vector V_1 is time-like and $a_j = -1$ then,

$$\begin{aligned} C_2(s) &= (\alpha - m_2 V_2)(s) \\ &= \alpha(s) - \left(\frac{\varepsilon_1 \varepsilon_2}{k_1(s)}\right) V_2(s) \\ &= \alpha(s) - \frac{1}{k_1(s)} V_2(s) \\ &= (0,0,0,0), \end{aligned}$$

where $\varepsilon_1 = \langle V_1, V_1 \rangle$.

Example 4.2. Let

$$\alpha(s) = (\cos s, \sin s, \cos 2s, \sin 2s),$$

then differentiating $\alpha(s)$ we get

$$E_1(s) = \alpha'(s) = (-\sin s, \cos s, -2\sin 2s, 2\cos 2s), \quad \|E_1(s)\| = \sqrt{5}$$

$$V_1(s) = \frac{E_1(s)}{\|E_1(s)\|} = \frac{1}{\sqrt{5}}(-\sin s, \cos s, -2\sin 2s, 2\cos 2s),$$

and

$$E_2(s) = \alpha''(s) - \langle \alpha''(s), V_1(s) \rangle V_1(s) = \alpha''(s), \quad \|E_2(s)\| = \sqrt{17},$$

$$V_2(s) = \frac{E_2(s)}{\|E_2(s)\|} = \frac{1}{\sqrt{17}}(-\cos s, -\sin s, -4\cos 2s, -4\sin 2s),$$

and

$$\begin{aligned} E_3(s) &= \alpha'''(s) - \langle \alpha'''(s), V_1(s) \rangle V_1(s) - \langle \alpha'''(s), V_2(s) \rangle V_2(s) \\ &= \frac{1}{5}(-12\sin s, 12\cos s, 6\sin 2s, -6\cos 2s), \quad \|E_3(s)\| = \frac{6\sqrt{5}}{5}, \end{aligned}$$

$$V_3(s) = \frac{E_3(s)}{\|E_3(s)\|} = \frac{1}{\sqrt{5}}(-2\sin s, 2\cos s, \sin 2s, -\cos 2s).$$

By using

$$k_i(s) = \frac{\|E_{i+1}(s)\|}{\|E_1(s)\|\,\|E_i(s)\|}$$

we obtain

$$k_1(s) = \frac{\sqrt{17}}{5}, \quad k_2(s) = \frac{6}{5\sqrt{17}}.$$

If the vector V_1 is time-like and $a_j = +1$,

$$\begin{aligned} C_2(s) &= (\alpha + m_2 V_2)(s) \\ &= \alpha(s) + \left(\frac{\varepsilon_1 \varepsilon_2}{k_1(s)}\right) V_2(s) \\ &= \frac{1}{17}(22\cos s, 22\sin s, 37\cos 2s, 37\sin 2s), \end{aligned}$$

and

$$\begin{aligned} C_3(s) &= (\alpha + m_2 V_2 + m_3 V_3)(s) \\ &= \alpha(s) + \left(\frac{\varepsilon_1 \varepsilon_2}{k_1(s)}\right) V_2(s) + \left(\frac{\varepsilon_1 \varepsilon_2}{k_1(s)}\right)' \frac{\varepsilon_3}{k_2(s)} V_3(s) \\ &= C_2(s). \end{aligned}$$

If the vector V_1 is time-like and $a_j = -1$,

$$\begin{aligned} C_2(s) &= (\alpha - m_2 V_2)(s) \\ &= \alpha(s) - \left(\frac{\varepsilon_1 \varepsilon_2}{k_1(s)}\right) V_2(s) \\ &= \frac{1}{17}(12\cos s, 12\sin s, -3\cos 2s, -3\sin 2s), \end{aligned}$$

and

$$\begin{aligned} C_3(s) &= (\alpha - m_2 V_2 - m_3 V_3)(s) \\ &= \alpha(s) - \left(\frac{\varepsilon_1 \varepsilon_2}{k_1(s)}\right) V_2(s) - \left(\frac{\varepsilon_1 \varepsilon_2}{k_1(s)}\right)' \frac{\varepsilon_3}{k_2(s)} V_3(s) \\ &= C_2(s), \end{aligned}$$

where $\left(\frac{1}{k_1(s)}\right)' = 0.$

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