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A GENERALIZATION OF PURELY EXTENDING MODULES RELATIVE TO A TORSION THEORY

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Abstract. In this work we introduce a new concept, namely, purely τ_s-extending modules (rings) which is torsion-theoretic analogues of extending modules and then we extend many results from extending modules to this new concept. For instance, we show that for any ring R with unit, ${}_RR$ is purely τ_s-extending if and only if every cyclic τ-nonsingular R-module is flat. Also, we make a classification for the direct sums of the rings to be purely τ_s-extending.

1. Introduction

Injective modules have been intensively studied in the 1960s and 1970s in module theory and more generally in algebra. As a generalization of injective modules, extending modules (CS), that is every closed submodule is a direct summand, have been studied widely in last three decades. In general setting, Chatters and Hajarnavis [7], Harmancı and Smith [23], Kamal and Muller [24] and their schools can be mentioned involving studies of extending modules.

Recently, torsion-theoretic analogues of extending modules has been studied on many results and concepts, such primarily studies as, Asgari and Haghany [4], Berktaş, Doğruöz and Tarhan [6], Crivei [11], Çeken and Alkan [12], Doğruöz [13]. Clark [8] defined a module M is *purely extending* if every submodule of M is essential in a pure submodule of M, equivalently every closed submodule of M is pure in M. A submodule K of a module M is *essential (in M)* if $N \cap K \neq 0$ for every non-zero submodule K of M. A submodule K of a module M is *closed (in M)* if K has no proper essential extension in M, i.e., whenever L is a submodule of M such that K is essential in L, then $K = L$. Al-Bahrani [1] generalized purely extending modules as a purely y-extending module using

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s-closed submodules which was defined by Goodearl [21] such as a submodule N of a module M is *s-closed* in M if M/N is nonsingular. So a module M is called *purely y-extending* if every s-closed submodule of M is pure in M. In fact, Al-Bahrani [1] belike misused the terminology of s-closed submodules. They used the term y-closed (purely y-extending) instead of s-closed (purely s-extending) respectively. In this study, we use s-closed submodule and purely s-extending module instead of y-closed submodule and purely y-extending module in the sense of Al-Bahrani [1].

We use the concept 'purity' in the sense of Cohn [10] (as in [8]) which implies definition of Anderson and Fuller [3], that is, a submodule N of an R-module M is called *pure submodule* in M in case $IN = N \cap IM$ for each finitely generated right ideal I of the ring R (see also [26]). In the present paper we introduce purely τ_s-extending modules and then we extend many results from [1], [8] and [21] to this new concept.

For instance, we show that:

Theorem 1: Let R be a τ-torsion ring and M be an R-module. Let $E(M)$ be an injective hull of M. Then M is a purely τ_s-extending module if and only if $A \cap M$ is pure in M for every direct summand A of $E(M)$ such that the submodule $A \cap M$ is τ_s-closed in M.

Proposition 5: Let R be a ring with identity. Then ${}_RR$ is purely τ_s-extending if and only if every cyclic τ-nonsingular R-module is flat.

and

Theorem 6: Let R be a commutative domain and every essential ideal of R is τ-dense in R. Then the following properties are equivalent:

- **(1):** R is a semi-hereditary ring.
- **(2):** $R \oplus R$ is an extending module.
- **(3):** $R \oplus R$ is a purely extending module.
- **(4):** $R \oplus R$ is a purely s-extending module.
- **(5):** $R \oplus R$ is a purely τ_s-extending module.
- **(6):** for each $n \in \mathbb{N}$, $\bigoplus_n R$ is an extending module.
- **(7):** for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely extending module.
- **(8):** for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely s-extending module.
- **(9):** for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely τ_s-extending module.

which is a torsion-theoretic analogue of [8, Proposition 1.6].

Throughout the work R will be an associative ring with identity and all R-modules will be unitary left R-modules unless otherwise stated. *R-Mod* will be the category of unitary left R-modules, and all modules and module homomorphisms will belong to *R-Mod*. By a *class* $\mathcal{X}$ of R-modules we mean a collection of R-modules containing the zero module and closed under isomorphism, i.e., any module which is isomorphic to some module in $\mathcal{X}$ also belongs to $\mathcal{X}$. If a submodule N of a module M belongs to $\mathcal{X}$ class, then N is called *$\mathcal{X}$-submodule* of M. The class of $\mathcal{X}$ closed under extension by short exact sequence we mean for a short exact sequence

$$0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$$

of R-modules A, B, C, if A and C are bought belong to the class of $\mathcal{X}$, then B is also belongs to $\mathcal{X}$ class.

Let $\tau := (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on R-Mod. The modules in $\mathcal{T}$ are called τ-*torsion* modules and the modules in $\mathcal{F}$ are called τ-*torsion-free* modules. Let $M \in R$-Mod. Then the τ-*torsion submodule* of M, denoted by $\tau(M)$, is defined to be the sum of all τ-torsion submodules of M. Thus $\tau(M)$ is the unique largest τ-torsion submodule of M and $\tau(M/\tau(M)) = 0$ for an R-module M. Also the module M is τ-torsion (resp. τ-torsion-free) if and only if $\tau(M) = M$ (resp. $\tau(M) = 0$). In our study, we mean a ring R is τ-torsion if ${}_RR$ is τ-torsion.

Let M be an R-module. A submodule N of M is called τ-*dense* in M if M/N is τ-torsion. A submodule N of M is called τ-*essential* in M denoted by ($N \leq_{\tau_e} M$) if N is essential in M and M/N is τ-torsion (see [19], originally defined by Tsai in 1965 [29]). Define the set $Z_\tau(M) = \{m \in M \mid Ann(m) \leq_{\tau_e} R\}$. Here $Z_\tau(M)$ is called the τ-*singular submodule* of M. Then the module M is called τ-*singular* if $Z_\tau(M) = M$ and τ-*nonsingular* if $Z_\tau(M) = 0$ ([20]). We mean $Z(M)$ the singular submodule of a module M which is consists of singular elements of M, i.e., elements annihilated by essential left ideals. The module M is singular (resp. nonsingular) if $Z(M) = M$ (resp. $Z(M) = 0$). For the singular and nonsingular notions (see also [21], [22]). If a ring R is τ-torsion, then every left ideal I of R is τ-dense in it, i.e., R/I is τ-torsion in the sense of [19]. Therefore, clearly $Z_\tau(M) = Z(M)$ over a τ-torsion ring R.

For elementary, additional and unexplained terminology the reader is referred to [3] or [30] for module and ring theory, [19] and [28] for torsion theory, [15] for extending modules and [26] for homological algebra.

2. Purely τ_s-Extending Modules

Definition 1. Let M be an R-module and N be a submodule of M. We call N is a τ_s-*closed submodule* of M if the factor module M/N is τ-nonsingular and it is denoted by $N \leq_{\tau_s c} M$.

Definition 2. Let M be an R-module. We call M is a *purely* τ_s-*extending module* if every τ_s-closed submodule of M is pure in M.

Lemma 1. *Let R be a τ-torsion ring. Then every τ_s-closed submodule of a module M is closed in M.*

Proof. Let N be a τ_s-closed submodule of M. Then the factor module M/N is τ-nonsingular i.e., $Z_\tau(M/N) = 0$. Since R is τ-torsion, clearly $Z_\tau(M/N) = Z(M/N)$. Assume N is not closed in M. Then there exists a submodule K of M such that K contains N as an essential submodule. So the factor module K/N

is singular [21]. Hence $Z(K/N) = K/N$. On the other hand, $Z(K/N) = 0$ since $Z(K/N)$ is a submodule of $Z(M/N)$. Hence K/N is nonsingular. But since K/N is singular, it must be zero (i.e $K/N = 0$). Therefore, $N = K$ and so N is closed submodule of M. □

Corollary 1. *Let R be a τ-torsion ring. Then every purely extending R-module is purely τ_s-extending.*

Proof. Let M be a purely extending module and N be a τ_s-closed submodule of M. Since R is τ-torsion N is closed in M by Lemma 1. From [8, Lemma 1.1] every closed submodule of M is pure in M. So N is pure in M. Therefore M is purely τ_s-extending module. □

As in general extending module theory we have some of the fundamental properties of purely τ_s-extending modules as follows:

Lemma 2. *Let $M = M_1 \oplus M_2$ be a purely τ_s-extending module. Then M_1 and M_2 are also purely τ_s-extending modules i.e., any direct summand of a purely τ_s-extending module is purely τ_s-extending.*

Proof. Let $M = M_1 \oplus M_2$ be a purely τ_s-extending module and let N_1 be a τ_s-closed submodule of M_1. Then $Z_\tau(M_1/N_1) = 0$. For the proof we want to show that N_1 is pure in M_1. First let us show that N_1 is τ_s-closed in M i.e., (M/N_1) is τ-nonsingular.

Assume M/N_1 is not τ-nonsingular module. Thus $Z_\tau(M/N_1) \neq 0$. Then there exists an element $N_1 \neq m + N_1 \in M/N_1$ such that $Ann(m + N_1) \leq_{\tau_e} R$. On the other hand, since $m \in M = M_1 \oplus M_2$, there exist $m_1 \in M_1$ and $m_2 \in M_2$ such that $m = m_1 + m_2$ and this writing unique. Thus

$$\begin{aligned} Ann(m + N_1) &= Ann((m_1 + m_2) + N_1) = Ann(m_1 + N_1 + m_2 + N_1) \\ &= Ann(m_1 + N_1) \cap Ann(m_2 + N_1) \end{aligned}$$

(see [3, Proposition 2.16]). In addition, since $Ann(m + N_1) \leq_{\tau_e} R$, we have $Ann(m_1 + N_1) \cap Ann(m_2 + N_1) \leq_{\tau_e} R$. Since $Ann(m_1 + N_1) \cap Ann(m_2 + N_1) \subseteq Ann(m_1 + N_1) \subseteq R$, we have $Ann(m_1 + N_1) \leq_{\tau_e} R$. But this contradicts with $Z_\tau(M/N_1) \neq 0$. Hence $Z_\tau(M/N_1) = 0$ i.e., N_1 is a τ_s-closed submodule of M. By the hypothesis N_1 is pure in M since M is purely τ_s-extending module. By [17, Proposition 1.2 (2)] N_1 is pure in M_1. Thus M_1 is purely τ_s-extending module. Similarly it can be shown that M_2 is also purely τ_s-extending module. □

Corollary 2. *Let $M = \bigoplus_{i \in I} M_i$ be a purely τ_s-extending module where I is a finite index set. Then for every $i \in I$, M_i is purely τ_s-extending.*

Proof. It is clear from Lemma 2. □

Lemma 3. *Let C be an R-module. Then C is a τ-nonsingular module if and only if $Hom_R(A, C) = 0$ for every τ-singular R-module A.*

Proof. Let $f : A \longrightarrow C$ be an R-module homomorphism where C is a τ-nonsingular module and A is a τ-singular R-module. Then $f(A) = f(Z_\tau(A))$. We show $f(Z_\tau(A)) \leq Z_\tau(C)$. If $x \in f(Z_\tau(A))$ then there is an element $a \in Z_\tau(A)$ such that $x = f(a)$. So $Ann(a) \leq_{\tau_e} R$. If $r \in Ann(a)$, then $rx = rf(a) = f(ra) = 0$ i.e., $r \in Ann(x)$. Since $Ann(a) \leq Ann(x) \leq R$, we have $Ann(x) \leq_{\tau_e} R$ i.e., $x \in Z_\tau(C)$. By the hypothesis, since $Z_\tau(C) = 0$, $f = 0$ and thus $Hom_R(A, C) = 0$.

For the converse let $Hom_R(A, C) = 0$ for every τ-nonsingular R-module A. Specially $Hom_R(Z_\tau(C), C) = 0$. So the inclusion map $Z_\tau(C) \longrightarrow C$ is zero. Hence $Z_\tau(C) = 0$ and so C is τ-nonsingular module. □

Lemma 4. *The class of τ-nonsingular modules is closed under extensions by short exact sequences.*

Proof. Let C and A be τ-nonsingular modules and consider the following short exact sequence

$$0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$$

For every τ-singular R-module M, using Lemma 3, we have $Hom_R(M, C) = 0$ and $Hom_R(M, A) = 0$. Then the following short exact sequence

$$0 \longrightarrow Hom_R(M, C) \longrightarrow Hom_R(M, B) \longrightarrow Hom_R(M, A) \longrightarrow 0$$

yields $Hom_R(M, B) = 0$. Again by Lemma 3 the R-module B must be τ-nonsingular. □

Next we can show τ_s-closed submodules have transitivity property.

Lemma 5. *Let M be an R-module and let K and N be submodules of M such that $K \leqslant N$. If K is τ_s-closed submodule of N and N is τ_s-closed submodule of M, then K is τ_s-closed submodule of M.*

Proof. Since K is τ_s-closed submodule of N and N is τ_s-closed submodule of M, $Z_\tau(N/K) = 0$ and $Z_\tau(M/N) = 0$. We must show that $Z_\tau(M/K) = 0$. Consider the following short exact sequence

$$0 \longrightarrow N/K \longrightarrow M/K \longrightarrow M/N \longrightarrow 0$$

By Lemma 4, the class of τ-nonsingular modules are closed under extensions by short exact sequences. Since N/K and M/N are both τ-nonsingular, M/K is τ-nonsingular. Hence $Z_\tau(M/K) = 0$. Thus K is τ_s-closed submodule of M. □

Now we have some basic properties as follows.

Lemma 6. *Any τ_s-closed submodule of a purely τ_s-extending module is purely τ_s-extending.*

Proof. Let M be a purely τ_s-extending module and let N be a τ_s-closed submodule of M. Then M/N is τ-nonsingular. Let K be a τ_s-closed submodule of N. Then by Lemma 5, K is a τ_s-closed submodule of M. Since M is purely τ_s-extending module, K is pure in M. By [17, Proposition 1.2 (2)], K is pure in N. So N is purely τ_s-extending module. □

There exist submodules K, L of a module M such that K and L both closed submodules of M but $K \cap L$ is not closed in K, L or M (see [21, Example 1.6]). But we have the following in our case.

Proposition 1. *Let M be an R-module and N, K be τ_s-closed submodules of M. Then $N \cap K$ is a τ_s-closed submodule of M.*

Proof. Let M be an R-module and N, K be τ_s-closed submodules of M. Then M/K and M/N are τ-nonsingular, i.e., $Z_\tau(M/N) = 0$ and $Z_\tau(M/K) = 0$. Assume $Z_\tau(M/(N \cap K)) \neq 0$. Then there is a $(N \cap K) \neq \bar{m} \in M/(N \cap K)$ such that $Ann(\bar{m}) \leq_{\tau_e} R$. Now for $\bar{m} = m + (N \cap K)$, $m \notin N \cap K$. On the other hand for $m \in M$, choose the elements $\hat{m} = m + N \in M/N$ and $\tilde{m} = m + K \in M/K$. Then we have $Ann(\bar{m}) \subseteq Ann(\hat{m})$ and $Ann(\bar{m}) \subseteq Ann(\tilde{m})$. Indeed, now let $0 \neq r \in Ann(\bar{m})$. Then $r\bar{m} = 0$ and so $rm + (N \cap K) = N \cap K$. Hence $rm \in N \cap K$. So we have $rm \in N$ and $rm \in K$. Thus $rm + N = N$ and $rm + K = K$, i.e. $r\hat{m} = 0$ and $r\tilde{m} = 0$. Consequently $r \in Ann(\hat{m})$ and $r \in Ann(\tilde{m})$. Hence $Ann(\bar{m}) \subseteq Ann(\hat{m})$ and $Ann(\bar{m}) \subseteq Ann(\tilde{m})$. On the other hand, since $Ann(\bar{m}) \leq_{\tau_e} R$ we have $Ann(\hat{m}) \leq_{\tau_e} R$ and $Ann(\tilde{m}) \leq_{\tau_e} R$. Then by hypothesis $Z_\tau(M/N) = 0$ and $Z_\tau(M/K) = 0$, we have $m \in N$ and $m \in K$ and so $m \in N \cap K$. Hence $\bar{m} = m + (N \cap K) = N \cap K$. This is a contradiction. Thus $Z_\tau(M/(N \cap K)) = 0$. Therefore, $N \cap K$ is a τ_s-closed submodule of M. □

Corollary 3. *Any intersection of τ_s-closed submodules is also τ_s-closed.*

Proof. It is an evident result of Proposition 1. □

Lemma 7. *Let M be an R-module and let K, L be submodules of M such that $K \leqslant L$. If L is a τ_s-closed submodule of M, then L/K is a τ_s-closed submodule of M/K.*

Proof. Let L be a τ_s-closed submodule of M. Then $Z_\tau(M/L) = 0$. On the other hand, $(M/K)/(L/K) \cong M/L$ and since τ-nonsingular modules are closed under isomorphisms, $Z_\tau((M/K)/(L/K)) = 0$. Hence L/K is τ_s-closed in M/K. □

Lemma 8. *Let M be an R-module and let K, L be submodules of M such that $K \leq L$. If the submodule L/K is τ_s-closed in M/K, then L is a τ_s-closed submodule of M.*

Proof. Since L/K is a τ_s-closed submodule of M/K, $Z_\tau((M/K)/(L/K)) = 0$. Since $(M/K)/(L/K) \cong M/L$ and τ-nonsingular modules are closed under isomorphisms, $Z_\tau(M/L) = 0$. Hence L is a τ_s-closed submodule of M. □

Proposition 2. *Let M be a purely τ_s-extending R-module and N be a τ_s-closed submodule of M. Then the factor module M/N is purely τ_s-extending.*

Proof. Let M be a purely τ_s-extending R-module and N be a τ_s-closed submodule of M. By the definition of purely τ_s-extending module, N is pure in M. For $N \leq K \leq M$ let K/N be τ_s-closed in M/N. Now $(M/N)/(K/N) \simeq M/K$ and since the τ-nonsingular modules are closed under isomorphisms, $Z_\tau(M/K) = 0$. So K is τ_s-closed submodule of M. Since M is purely τ_s-extending, K is pure in M. By [17, Proposition 1.2 (3)] K/N is pure in M/N. Thus M/N is purely τ_s-extending. □

Let M be an R-module. For an arbitrary submodule N of M by Zorn's Lemma there is a submodule K of M maximal with respect to N is essential in K. The submodule K is called *closure* of N in M ([27]). See also [14] for torsion theoretic version of closures.

Now we give another generalization of closures relative to a torsion theory as follows:

Definition 3. Let M be an R-module and let N be a submodule of M. The smallest τ_s-closed submodule K of M which is containing N is called τ_s-*closure* of N in M. The τ_s-closure of N is denoted by $N^{-\tau_s}$.

Lemma 9. *Every submodule N of an R-module M has a τ_s-closure in M.*

Proof. Let M be an R-module and N be a submodule of M. Now define the set $\mathcal{S} = \{K \leq M \mid N \subseteq K \text{ and } K \leq_{\tau_s c} M\}$. Since $Z_\tau(M/M) = 0$, M is τ_s-closed in M and so $M \in \mathcal{S}$. Then $\mathcal{S}$ is non-empty. Let $\mathcal{C}$ be a chain in $\mathcal{S}$. Take $C = \bigcap_{K_i \in \mathcal{C}} K_i$. By Corollary 3 C is a τ_s-closed submodule of M. Then $C \in \mathcal{S}$. By Zorn's Lemma there is a minimal element in $\mathcal{S}$. If we call this element such as H then H is τ_s-closure of N in M. Thus every submodule N of M has a τ_s-closure in M. □

Proposition 3. *An R-module M is a purely τ_s-extending if and only if the τ_s-closure of N (i.e., $N^{-\tau_s}$) is pure in M for every submodule N of M.*

Proof. Let M be a purely τ_s-extending module. Then every τ_s-closed submodule of M is pure in M. By Zorn's Lemma every submodule N of M has a τ_s-closure in M. By the definition of τ_s-closure, the submodule $N^{-\tau_s}$ is τ_s-closed in M and by the hypothesis the submodule $N^{-\tau_s}$ is pure in M.

Conversely, let K be a τ_s-closed submodule in M. By the definition of τ_s-closure, $K^{-\tau_s} = K$. By the hypothesis $K^{-\tau_s}$ i.e. K is a pure submodule in M. Then any τ_s-closed submodule of M is pure in M. Thus M is a purely τ_s-extending module. □

Theorem 1. *Let R be a τ-torsion ring, let M be an R-module and $E(M)$ be the injective hull of M. Then, M is a purely τ_s-extending module if and only if $A \cap M$ is pure in M for every direct summand A of $E(M)$ such that the submodule $A \cap M$ is τ_s-closed in M.*

Proof. Let R be a τ-torsion ring, M be an R-module, $E(M)$ be the injective hull of M and M be a purely τ_s-extending module. Then for every direct summand A of $E(M)$ such that $A \cap M$ is a τ_s-closed submodule of M it is clear that $A \cap M$ is pure in M.

Conversely, let A be a τ_s-closed submodule of M and let B be a complement of A in M. Then $A \oplus B$ is essential in M [21, Proposition 1.3]. Now it is clear that $A \oplus B$ is essential in $E(M)$. Hence $E(A) \oplus E(B) = E(A \oplus B) = E(M)$ [22]. Since $A = A \cap M \leq_e E(A) \cap M$, $(E(A) \cap M)/A$ is singular (see [21]). Moreover, since R is τ-torsion ring $(E(A) \cap M)/A$ is τ-singular. On the other hand since $(E(A) \cap M)/A \leq M/A$ and A is τ_s-closed submodule of M, M/A is τ-nonsingular and thus $(E(A) \cap M)/A$ is τ-nonsingular. Therefore, $(E(A) \cap M)/A = 0$ and so $E(A) \cap M = A$. Since A is τ_s-closed in M, $E(A) \cap M$ is also τ_s-closed in M. Since $E(A)$ is a direct summand of $E(M)$ by the hypothesis $E(A) \cap M$ is a pure submodule of M. Hence A is pure in M. Thus M is a purely τ_s-extending module. □

Theorem 2. *Let R be a τ-torsion ring, let M be an R-module and let $E(M)$ be the injective hull of M. Assume $A+M$ be a flat module for every direct summand A of $E(M)$ with $A \cap M$ is τ_s-closed submodule of M. Then M is a purely τ_s-extending module.*

Proof. Let A be a direct summand of $E(M)$ such that $A \cap M$ is τ_s-closed in M. Consider the following short exact sequences of R-modules

$$0 \longrightarrow A \cap M \xrightarrow{i_1} M \xrightarrow{f_1} M/(A \cap M) \longrightarrow 0$$

and

$$0 \longrightarrow A \xrightarrow{i_2} A + M \xrightarrow{f_2} (A+M)/A \longrightarrow 0$$

where i_1, i_2 are inclusion maps and f_1, f_2 are natural epimorphisms. Since A is a direct summand of $E(M)$, there is a submodule $A^{'}$ of $E(M)$ such that $E(M) = A \oplus A^{'}$. Thus A is also a direct summand of $A+M$ such as $A+M = (A+M) \cap E(M) = (A+M) \cap (A \oplus A^{'}) = A \oplus ((A+M) \cap A^{'})$. Here $((A+M) \cap A^{'})$ is flat as a direct summand of a flat module $A+M$. Since $(A+M)/A \cong ((A+M) \cap A^{'})$, $(A+M)/A$ is flat. On the other hand, the factor module $M/(A \cap M)$ is again flat since $M/(A \cap M) \cong (A+M)/A$. By [17, Theorem 1.7] $A \cap M$ is pure in M. Hence by Theorem 1, M is a purely τ_s-extending module. □

3. Purely τ_s-Extending Rings

If the ring R is purely τ_s-extending as an R-module over itself then R is called *purely τ_s-extending.*

A (von Neumann) regular ring R as an R-module over itself, i.e., ${}_RR$ can be given an example of purely τ_s-extending ring since every left ideal is pure in it by [17, Theorem 2.1].

Fieldhouse in [17] generalizing (von Neumann) regular ring and define, for any ring R, an R-module M is called (von Neumann) *regular* if all its submodules are pure in M.

Therefore, since all (left) R-modules over a (von Neumann) regular ring is regular by [17, Theorem 3.1], thus all R-modules over a (von Neumann) regular ring R is purely τ_s-extending. Also any regular module over any ring R can be given as an example of purely τ_s-extending modules.

3.1. Multiplication Modules.

Let R be a commutative ring and M be an R-module. For every submodule N of M if there exists an ideal I of R such that $N = IM$, then M is called a *multiplication module.* For every submodule N of M let us define

$$(N : M) = \{r \in R \mid rM \subseteq N\}.$$

Then M is an multiplication R-module if and only if $N = (N : M)M$ ([5]).

Definition 4. *[9]* Let M be an R-module and N be a submodule of M. If

$$N = Hom(M, N)N = \Sigma\{\varphi(N) \mid \varphi : M \to N\}$$

then N is called *an idempotent submodule* of M. If every submodule of M is idempotent, then M is called a *fully idempotent module*.

Theorem 3. *[16, Teorem 2.11] Let M be a multiplication R-module and $M = M_1 \oplus M_2$, is a direct sum of fully idempotent submodules M_1 and M_2. Then M is a fully idempotent module.*

Lemma 10. *[16, Lemma 2.13] Let M be a fully idempotent R-module, N be a submodule of M and I be an ideal of R. Then $N \cap MI = NI$, i.e., N is pure in M.*

Now we can give the following teorem by using fully idempotent submodules:

Theorem 4. *Let R be a commutative ring and let $M = M_1 \oplus M_2$ be a multiplication R-module with fully idempotent submodules M_1, M_2 of M. Then M is a purely τ_s-extending module.*

Proof. Let M be a multiplication R-module and N be a τ_s-closed submodule of M. By Teorem 3 M is fully idempotent R-module and by Lemma 10 the τ_s-closed submodule N of M is pure in M. Hence M is purely τ_s-extending. □

Now we can give a characterization of a purely τ_s-extending R-module with a ring as follows:

Proposition 4. *Let R be a commutative ring and let M be a faithful multiplication R-module. If ${}_RR$ is purely τ_s-extending module then M is also purely τ_s-extending module.*

Proof. Let N be a τ_s-closed submodule of M. Since M is multiplication R-module, we can write $N = (N : M)M$. Claim: $(N : M)$ is τ_s-closed submodule in ${}_RR$. Assume $(N : M)$ is not τ_s-closed in R. Then $R/(N : M)$ is not τ-nonsingular that is, $Z_\tau(R/(N : M)) \neq 0$. Then there exists at least one non-zero element $\bar{r}$ of $R/(N : M)$ such that $Ann(r + (N : M))$ is τ-essential in R. So $\bar{r} = r + (N : M) \neq (N : M)$. Then there is an element $0 \neq m_0 \in M$ such that $rm_0 \notin N$. Now $Ann(r + (N : M)) \subseteq Ann(rm_0 + N)$. If $s \in Ann(r + (N : M))$, then $sr + (N : M) = (N : M)$. Hence we have $sr \in (N : M)$ so it is easy to check that $(sr)M \subseteq N$ (*). Let us show that $s \in Ann(rm_0+N)$. Now $s(rm_0+N) = srm_0+N$ but since $(sr)M \subseteq N$ and by (*) for $m_0 \in M$, $srm_0 \in N$, i.e., $srm_0 + N = N$. So $s \in Ann(rm_0 + N)$. Hence we have $Ann(r + (N : M)) \subseteq Ann(rm_0 + N)$. On the other hand, since N is τ_s-closed in M it is clear that M/N τ-nonsingular. So $rm_0 + N = N$ but it contradicts with $rm_0 \notin N$. Hence $(N : M)$ must be τ_s-closed in R. Moreover since ${}_RR$ is purely τ_s-extending, $(N : M)$ is pure in R, i.e., $I(N : M) = IR \cap (N : M)$ for every finitely generated ideal I of R. Thus $I(N : M) = IR \cap (N : M) = I \cap (N : M)$. Therefore, by $N = (N : M)M$ we write $IN = I(N : M)M = (I \cap (N : M))M$. On the other hand, the equality $(I \cap (N : M))M = IM \cap (N : M)M$ holds since R is a commutative ring and M is a faithful multiplication R-module by applying [2, Proposition 1.6 (i)].

Now for the finitely generated ideal I of R, we have
$IN = I(N : M)M = (I \cap (N : M))M = IM \cap (N : M)M = IM \cap N$ ([5]). Therefore, the τ_s-closed submodule N of M is pure in M. Hence M is a purely τ_s-extending module. □

Remark 1. *[26, Proposition 3.46] Let R be an arbitrary ring. The left R-module R is a flat left R-module.*

In the sequel we use the flat ring in the sense of Rotman [26, Proposition 3.46], i.e the ring R is flat if ${}_RR$ is flat.

Proposition 5. *Let R be an arbitrary ring. Then ${}_RR$ is purely τ_s-extending if and only if every cyclic τ-nonsingular R-module is flat.*

Proof. Let ${}_RR$ be a purely τ_s-extending module. Let $M = Ra$ be a cyclic τ-nonsingular R-module which is generated by a. Define the map $f : R \to M$ with $f(r) = ra$. Clearly f is an epimorphism and $\mathrm{Ker}(f) = Ann(a)$. So $R/\mathrm{Ker}(f) = R/Ann(a) \cong Ra$. Moreover, since Ra is a τ-nonsingular module and the class of τ- nonsingular modules is closed under isomorphisms $R/Ann(a)$ is τ-nonsingular. Hence $Ann(a)$ is τ_s-closed in R. By the hypothesis $Ann(a)$ is pure in R. Since R is flat and $Ann(a)$ is pure in R, $R/Ann(a)$ is flat by [3, Lemma 19.18]. Therefore, Ra is flat.

Conversely, let K be a τ_s-closed ideal of R. Then R/K is τ-nonsingular. By the hypothesis R/K is flat as a left R-module. Thus by [3, Lemma 19.18], K is pure in R. Thus ${}_RR$ is a purely τ_s-extending. □

Theorem 5. *Let R be a ring. Then $R \oplus R$ is purely τ_s-extending if and only if every τ-nonsingular 2-generated R-module is flat.*

Proof. Let $M = Rm_1 + Rm_2$ be a τ-nonsingular R-module. Define the map $f : R \oplus R \to M$ with $f(r_1, r_2) = r_1 m_1 + r_2 m_2$. Now it is clear that f is an epimorphism. Hence $(R \oplus R)/\mathrm{Ker}(f) \cong M$. Since $(R \oplus R)/\mathrm{Ker}(f)$ is τ-nonsingular, $\mathrm{Ker}(f)$ is a τ_s-closed submodule of $R \oplus R$. By the hypothesis $\mathrm{Ker}(f)$ is pure in $R \oplus R$. Since R is is flat as an R-module, $R \oplus R$ is flat ([21]). Thus by [17, Proposition 1.3 (3)], we have the R-module M is flat.

For the converse, let C be a τ_s-closed submodule of $R \oplus R$. Then $(R \oplus R)/C$ is τ-nonsingular. On the other hand, since $R \oplus R$ is a 2-generated R-module, $(R \oplus R)/C$ is also a 2-generated τ-nonsingular R-module. By the hypothesis $(R \oplus R)/C$ is flat. Then by [17, Theorem 1.7] we get C is pure in $R \oplus R$. Thus $R \oplus R$ is purely τ_s-extending. □

Corollary 4. *Let R be a ring and I be a finite index set. Then $\oplus_I R$ is purely τ_s-extending if and only if every τ-nonsingular I-generated R-module is flat.*

3.2. **Semi-hereditary Rings.** Let R be a ring with unit element. If every left (right) ideal of R is projective then R is called a left (right) *hereditary ring* . If every finitely generated left (right) ideal of R is projective then R is called a left (right) *semi-hereditary ring* ([28]). A module M over a commutative domain R is said to be *torsion-free* if for $m \in M$ and $r \in R$, $rm = 0 \Rightarrow r = 0$ or $m = 0$ [25].

Now we can give the following generalized characterization of purely τ_s-extending modules.

Theorem 6. *Let R be a commutative domain and every essential ideal of R is τ-dense in R. Then the following properties are equivalent:*

- **(1):** *R is a semi-hereditary ring.*
- **(2):** *$R \oplus R$ is an extending module.*
- **(3):** *$R \oplus R$ is a purely extending module.*
- **(4):** *$R \oplus R$ is a purely s-extending module.*
- **(5):** *$R \oplus R$ is a purely τ_s-extending module.*
- **(6):** *for each $n \in \mathbb{N}$, $\bigoplus_n R$ is an extending module.*
- **(7):** *for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely extending module.*
- **(8):** *for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely s-extending module.*
- **(9):** *for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely τ_s-extending module.*

Proof. The equivalence of (1), (2) and (6) are given in [15, Corollary 12.10].

In addition the equivalence of (1), (2), (3), (6) and (7) are given in [8, Proposition 1.6].

(3) $\Leftrightarrow$ (4). Every s-closed submodule of a module M is closed in M. But converse is true if M is nonsingular [21, Proposition 2.4]. Here since R is commutative domain, R is nonsingular. Therefore, the notion of closed submodule and s-closed submodule coincide. Thus the proof is clear by [8, Lemma 1.1] in fact, Lemma 1.1 is originally given by Fuchs [18].

(7) $\Leftrightarrow$ (8). It can be easily checked be like (3) $\Leftrightarrow$ (4).

(5) $\Rightarrow$ (4). Let K be a s-closed submodule of $R \oplus R$. Then $(R \oplus R)/K$ is nonsingular. Since any nonsingular module is τ-nonsingular. $(R \oplus R)/K$ is a τ-nonsingular. By the hypothesis K is pure in $R \oplus R$. Hence $R \oplus R$ is a purely s-extending module.

The implication of (9) $\Rightarrow$ (8) is a generalization of (5) $\Rightarrow$ (4).

(1) $\Rightarrow$ (5). Let K be a τ_s-closed submodule of $R \oplus R$. Then $(R \oplus R)/K$ is τ-nonsingular. Claim that $(R \oplus R)/K$ is torsion-free R-module. For this fact, let us assume $\overline{m}.r = \overline{0}$ and $r \neq 0$ for $\overline{m} \in (R \oplus R)/K$ and $r \in R$. Here $0 \neq r \in Ann(\overline{m})$. Thus $Ann(\overline{m}) \neq 0$. Since also R is a commutative domain, then all non-zero ideals of R are essential [25, 7.6]. Thus $Ann(\overline{m})$ is essential ideal in R. By hypothesis of the theorem, $Ann(\overline{m})$ is τ-dense in R. Thus $Ann(\overline{m}) \leq_{\tau_e} R$ and so, $\overline{m} \in Z_\tau((R \oplus R)/K)$. In this case, $\overline{m} = 0$ since $(R \oplus R)/K$ is τ-nonsingular. Therefore $(R \oplus R)/K$ is torsion-free. Thus applying [25, Collary 2.31] $(R \oplus R)/K$ is projective since $(R \oplus R)/K$ is 2-generated over the Prüfer domain R. So $(R \oplus R)/K$ is flat by [26, Proposition 3.46]. Thus K is pure in $R \oplus R$ by [17, Proposition 1.3]. Hence $R \oplus R$ is a purely τ_s-extending module

(1) $\Rightarrow$ (9) is also similar to (1) $\Rightarrow$ (5). This completes the proof.

In fact, the proof can be also completed by the following implications.

(4) $\Rightarrow$ (5). Let K be a τ_s-closed submodule of $R \oplus R$. Then $(R \oplus R)/K$ is τ-nonsingular, i.e., $Z_\tau((R \oplus R)/K) = 0$. By assumption, since R is a ring with essential ideal of R is τ-dense in it, τ-nonsingular and nonsingular modules are coincide. Therefore $(R \oplus R)/K$ is nonsingular and so K is s-closed in $R \oplus R$. By hypothesis, K is pure in $R \oplus R$. Therefore, $R \oplus R$ is purely τ_s-extending module.

(8) $\Rightarrow$ (9) is also similar to (4) $\Rightarrow$ (5). □

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