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ON APPROXIMATION BY BALAZS-STANCU TYPE RATIONAL FUNCTIONS

Ç. ATAKUT

Department of Mathematics, Faculty of Science, Ankara University, Ankara, TURKEY

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ABSTRACT

In this paper the author define Balazs-Stancu type rational functions and prove the approximation theorems for them.

1. INTRODUCTION

Bernstein polynomials play an important role in approximation theory and in other fields of mathematics. The classical Bernstein polynomials are, as well-known, the following

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \quad (n=1,2,...). \tag{1}$$

It is also known that if $f(x)$ is continuous in the interval $[0,1]$, these polynomials converge uniformly to $f(x)$. Later, Stancu [2] defined the following generalization of Bernstein polynomials:

$$B_n^{(\alpha,\beta)}(f,x)=\sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k}x^k(1-x)^{n-k} \quad (\alpha,\beta\geq 0) \tag{2}$$

and proved the theorem about its convergence to function f.

K. Balazs[1] introduced and considered some approximation properties of Bernstein type rational functions

$$R_n(f,x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k f\left(\frac{k}{b_n}\right), \quad (n=1,2,...) \tag{3}$$

and proved that if f is continuous in $[0,\infty)$, $f(x)=O(e^{\gamma x})(x\to\infty)$ with some γ then in any interval $[0,A]$ $(A>0)$ the estimate

$$\left|f(x)-R_n(f,x)\right|\leq C_o\omega_{2A}\left(n^{-1/3}\right) \quad (0\leq x\leq A) \tag{4}$$

holds for sufficiently large n's provided $a_n = n^{-1/3}, b_n = n^{2/3}$. Here C_o depends only on A and α, and $\omega_{2A}(f,\delta)$ is the modulus of continuity of f on the interval $[0,2A]$, that is $\omega_{2A}(f,\delta) = \sup\{|f(t) - f(x)| : t, x \in [0,2A], |t - x| \le \delta\}$. As it was noted in [1], the convergence of $R_n(f,x)$ holds under the more general conditions $a_n = \frac{b_n}{n} \to 0, \quad b_n \to \infty \, (n \to \infty)$ as well.

In this work, we define a Stancu type generalization of Balazs type rational functions in the form

$$R_n^{(\alpha,\beta)}(f,x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k f\left(\frac{k+\alpha}{b_n+\beta}\right), (\alpha,\beta \ge 0)\,(n=1,2,...)\,. \tag{5}$$

We shall prove convergence theorems for them. Moreover, we shall prove an asymptotic approximation theorem and show that the derivatives of Balazs-Stancu type rational functions also convergence to the derivative of the function.

In order to establish the approximation theorems for (5) we need the following results.

Lemma 1. [1], For $x \ge 0$, then the following identities hold:

$$\frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k = 1 \qquad (n = 1,2,...), \tag{6}$$

$$\frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k (k - b_n x) = -\frac{a_n b_n x^2}{1+a_n x}, \tag{7}$$

$$\frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k (k - b_n x)^2 = \frac{a_n^2 b_n^2 x^4 + b_n x}{(1+a_n x)^2} \tag{8}$$

where $a_n = \frac{b_n}{n}$ and $b_n > 0$ is an arbitrary real number.

(In what follows c_i, $i = 0,1,2,...$ will denote constants independent of n).

Lemma 2. If $x \ge 0$, then the inequality

$$A_n = \frac{1}{(1+a_n x)^n} \sum_{\left|\frac{k+\alpha}{b_n+\beta} - x\right| \ge \delta} e^{\gamma\left(\frac{k+\alpha}{b_n+\beta}\right)} \binom{n}{k} (a_n x)^k \le$$
$$\le \frac{c_2}{(b_n+\beta)^2} \left\{ \frac{a_n^2 b_n^2 x^4 + b_n x}{(1+a_n x)^2} + \frac{2(\beta x - \alpha) a_n b_n x^2}{1+a_n x} + (\alpha - \beta x)^2 \right\} \tag{9}$$

holds for sufficiently large n where $\delta > 0$ and γ are arbitrary fixed real numbers.

$a_n = \frac{b_n}{n} \to 0, \quad b_n \to \infty$ as $n \to \infty$.

Proof. By the Lagrange's theorem

$$e^{\frac{\gamma}{b_n+\beta}}-1=\frac{\gamma}{b_n+\beta}e^{\frac{\theta\gamma}{b_n+\beta}}\le\frac{\gamma}{b_n+\beta}e^{\frac{\gamma}{b_n+\beta}}\le c_o\frac{\gamma}{b_n+\beta}$$

for some $0<\theta<1$ as γ is fixed and $b_n\to\infty$, $\beta\ge 0$.
Also,

$$\left(\frac{1+a_nxe^{\frac{\gamma}{b_n+\beta}}}{1+a_nx}\right)^n=\left[\frac{1+a_nx+a_nx(e^{\frac{\gamma}{b_n+\beta}}-1)}{1+a_nx}\right]^n\le\left[1+\frac{b_nxc_o\frac{\gamma}{b_n+\beta}}{n(1+a_nx)}\right]^n\le e^{c_1x} \quad (10)$$

where $a_n=\frac{b_n}{n}$.

With the notation $t=xe^{\frac{\gamma}{b_n+\beta}}$ we have, if $\left|\frac{k+\alpha}{b_n+\beta}-x\right|\ge\delta$,

$$\begin{aligned}\left|\frac{k+\alpha}{b_n+\beta}-t\right|&=\left|\frac{k+\alpha}{b_n+\beta}-xe^{\frac{\gamma}{b_n+\beta}}\right|=\left|\frac{k+\alpha}{b_n+\beta}-x+x(1-e^{\frac{\gamma}{b_n+\beta}})\right|\\&\ge\left|\frac{k+\alpha}{b_n+\beta}-x\right|-|x|\left|1-e^{\frac{\gamma}{b_n+\beta}}\right|\ge\delta-|x|\left|1-e^{\frac{\gamma}{b_n+\beta}}\right|\ge\delta^*\end{aligned} \quad (11)$$

for sufficiently large n, where $\delta^*>0$ is constant. If $\left|\frac{k+\alpha}{b_n+\beta}-x\right|\ge\delta$, then (11) gives that

$$\frac{((k+\alpha)-t(b_n+\beta))^2}{(b_n+\beta)^2\delta^{*2}}\ge 1. \quad (12)$$

Using (10), (12) and summing for all k, the inequality

$$\begin{aligned}A_n&=\frac{1}{(1+a_nx)^n}\sum_{\left|\frac{k+\alpha}{b_n+\beta}-x\right|\ge\delta}e^{\gamma\left(\frac{k+\alpha}{b_n+\beta}\right)}\binom{n}{k}(a_nx)^k\\&=e^{\frac{\gamma\alpha}{b_n+\beta}}\left(\frac{1+a_nxe^{\frac{\gamma}{b_n+\beta}}}{1+a_nx}\right)^n\frac{1}{\left(1+a_nxe^{\frac{\gamma}{b_n+\beta}}\right)^n}\sum_{\left|\frac{k+\alpha}{b_n+\beta}-x\right|\ge\delta}\binom{n}{k}\left(a_nxe^{\frac{\gamma}{b_n+\beta}}\right)^k \quad (13)\\&\le e^{\frac{\gamma\alpha}{b_n+\beta}}\frac{e^{c_1x}}{(b_n+\beta)^2\delta^{*2}}\frac{1}{(1+a_nt)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nt)^k((k-b_nt)+(\alpha-\beta t))^2\end{aligned}$$

holds.

Using (13) and (8), applying $t = xe^{\frac{\gamma}{b_n+\beta}}$, where $\frac{\gamma}{b_n+\beta} \to 0$, if $n \to \infty$, we get

$$A_n \le \frac{c_2}{(b_n+\beta)^2}\left\{\frac{a_n^2 b_n^2 x^4 + b_n x}{(1+a_n x)^2} + \frac{2(\beta x-\alpha)a_n b_n x^2}{1+a_n x} + (\alpha-\beta x)^2\right\},$$

which proves the lemma.

Corollary 2. $\lim_{n\to\infty} A_n = 0$.

It is well-known [3] that if λ and δ are arbitrary positive values, then

$$\omega_{2A}(\lambda\delta) \le \omega_{2A}(\delta)(\lambda+1). \tag{14}$$

We now give the following convergence theorem.

Theorem 3. Let $f(x)$ be a continuous defined in $[0,\infty)$ such that $f(x) = O(e^{\gamma x})(x \to \infty)$, for some real number γ. Then in any interval $0 \le x \le A$ $(A \ge 0)$ the inequality

$$\left|f(x) - R_n^{(\alpha,\beta)}(f,x)\right| \le \left[\frac{\left(\left(n^{1/3}A^2 + (\beta A-\alpha)\right)^2 + n^{2/3}A\right)^{1/2}}{n^{2/3}+\beta}\right] + \frac{c_7}{\left(n^{2/3}+\beta\right)^2}\left[n^{2/3}(A^4+A) + 2A^2(\beta A-\alpha)n^{1/3} + (\alpha-\beta A)^2\right] \tag{15}$$

is valid for sufficiently large n.

Proof. Consider the difference $\Delta_n = \left|f(x) - R_n^{(\alpha,\beta)}(f,x)\right|$. By (5) and (6) we get

$$\Delta_n(f,x) \le \frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}\left|f(x) - f\left(\frac{k+\alpha}{b_n+\beta}\right)\right|\binom{n}{k}(a_n x)^k$$
$$\le \frac{1}{(1+a_n x)^n}\left(\sum_{\frac{k+\alpha}{b_n+\beta}\le 2A} + \sum_{\frac{k+\alpha}{b_n+\beta}>2A}\right) = S_1 + S_2. \tag{16}$$

We obtain by (14)

$$\left|f(x) - f\left(\frac{k+\alpha}{b_n+\beta}\right)\right| \le \omega_{2A}\left(\left|x - \frac{k+\alpha}{b_n+\beta}\right|\right) = \omega_{2A}\left(\delta_n \frac{1}{\delta_n}\left|x - \frac{k+\alpha}{b_n+\beta}\right|\right)$$
$$\le \omega_{2A}(\delta_n)\left(\frac{1}{\delta_n}\left|x - \frac{k+\alpha}{b_n+\beta}\right| + 1\right). \tag{17}$$

Using (16), (17) and (6)

$$S_1 \le \omega_{2A}(\delta_n)\frac{1}{\delta_n(1+a_nx)^n}\sum_{\frac{k+\alpha}{b_n+\beta}\le 2A}\left|x-\frac{k+\alpha}{b_n+\beta}\right|\binom{n}{k}(a_nx)^k+\omega_{2A}(\delta_n) \tag{18}$$
$$= S_1' + \omega_{2A}(\delta_n).$$

Using the Schwarz inequality, then considering (6) and (8) we obtain

$$S_1' \le \omega_{2A}(\delta_n)\frac{1}{\delta_n}\left\{\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\left(x-\frac{k+\alpha}{b_n+\beta}\right)^2\binom{n}{k}(a_nx)^k \times \frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k\right\}^{1/2}$$
$$\le \omega_{2A}(\delta_n)\frac{1}{\delta_n}\left[\frac{a_n^2b_n^2x^4+b_nx}{(b_n+\beta)^2(1+a_nx)^2}+\frac{2(\beta x-\alpha)a_nb_nx^2}{(b_n+\beta)^2(1+a_nx)}+\frac{(\beta x-\alpha)^2}{(b_n+\beta)^2}\right]^{1/2}. \tag{19}$$

Since $a_n = \frac{b_n}{n}$ and $b_n = n^{2/3}$, then by (18) and (19) we have

$$S_1 \le \omega_{2A}(\delta_n)\left\{\frac{1}{\delta_n}\frac{1}{n^{2/3}+\beta}\left[\left(n^{1/3}A^2+(\beta A-\alpha)\right)^{1/2}+n^{2/3}A\right]^{1/2}+1\right\}. \tag{20}$$

Chosing $\delta_n = \frac{\left[\left(n^{1/3}A^2+(\beta A-\alpha)\right)^{1/2}+n^{2/3}A\right]^{1/2}}{n^{2/3}+\beta}$, we get

$$S_1 \le \omega_{2A}\left[\frac{\left(\left(n^{1/3}A^2+(\beta A-\alpha)\right)^{1/2}+n^{2/3}A\right)^{1/2}}{n^{2/3}+\beta}\right]. \tag{21}$$

Since $f(x) = O(e^{\gamma x})(x \to \infty, \gamma \text{ fixed})$, the estimation of S_2 is an easy consequence of Lemma 2, if δ is chosen small enough:

$$S_2 = \sum_{\frac{k+\alpha}{b_n+\beta}>2A}\binom{n}{k}(a_nx)^k\left|f(x)-f\left(\frac{k+\alpha}{b_n+\beta}\right)\right|$$
$$\le \frac{1}{(1+a_nx)^n}\sum_{\frac{k+\alpha}{b_n+\beta}>2A}\binom{n}{k}(a_nx)^k c_5 e^{\gamma\left(\frac{k+\alpha}{b_n+\beta}\right)}$$
$$\le \frac{c_6}{(1+a_nx)^n}\sum_{\left|\frac{k+\alpha}{b_n+\beta}-x\right|\ge\delta}\binom{n}{k}(a_nx)^k e^{\gamma\left(\frac{k+\alpha}{b_n+\beta}\right)}$$
$$\le \frac{c_7}{(n^{2/3}+\beta)^2}\left[n^{2/3}(x^4+x)+2(\beta x-\alpha)x^2n^{1/3}+(\alpha-\beta x)^2\right].$$

Since $0 \le x \le A$, then we can obtain S_2 in the following way

$$S_2 \le \frac{c_7}{(n^{2/3}+\beta)^2}\left[n^{2/3}(A^4+A)+2(\beta A-\alpha)A^2 n^{1/3}+(\alpha-\beta A)^2\right]. \quad (22)$$

Finally, using (21) and (22), the inequality (16) may be written in the following way

$$\Delta_n(f,x) \le \omega_{2A}\left[\frac{\left(n^{2/3}A^2+(\beta A-\alpha)+n^{2/3}A\right)^{1/2}}{n^{2/3}+\beta}\right]$$

$$+\frac{c_7}{(n^{2/3}+\beta)^2}\left[n^{2/3}(A^4+A)+2(\beta A-\alpha)A^2 n^{1/3}+(\alpha-\beta A)^2\right], \ (0 \le x \le A).$$

This establishes the proof of Theorem 3.

Now, we prove an asymptotic approximation theorem for Balazs-Stancu type rational functions.

Theorem 4. Let $f(t)$ be a function defined in $[0,\infty)$ such that $f(t)=O(e^{\gamma t})(t\to\infty, \gamma$ is a fixed real number). If $f(t)$ has a finite second derivative at each point $t=x$, then

$$R_n^{(\alpha,\beta)}(f,x)=f(x)+a_n f'(x)\left(-\frac{b_n x^2}{(b_n+\beta)(1+a_n x)}+\frac{(\alpha-\beta x)}{a_n(b_n+\beta)}\right)$$

$$+a_n f''(x)\left(\frac{a_n b_n^2 x^4+\frac{b_n}{a_n}x}{2(b_n+\beta)^2(1+a_n x)^2}-\frac{(\alpha-\beta x)b_n x^2}{(b_n+\beta)^2(1+a_n x)}+\frac{(\alpha-\beta x)^2}{2a_n(b_n+\beta)^2}\right)+a_n\rho_n \quad (23)$$

where $\rho_n\to 0$, $a_n=\frac{b_n}{n}\to 0$ and $\frac{n^{1/2}}{b_n}\to 0$ as $n\to\infty$.

Proof. By the conditions of the theorem, $f''(x)$ is finite, thus we may write

$$f(t)=f(x)+f'(x)(t-x)+\left(\frac{f''(x)}{2}+\lambda(t)\right)(t-x)^2 \quad (24)$$

where $\lambda(t)\to 0$ as $t\to x$. Hence,

$$f\left(\frac{k+\alpha}{b_n+\beta}\right)=f(x)+f'(x)\left(\frac{k+\alpha}{b_n+\beta}-x\right)+\left[\frac{f''(x)}{2}+\lambda\left(\frac{k+\alpha}{b_n+\beta}\right)\right]\left(\frac{k+\alpha}{b_n+\beta}-x\right)^2. \quad (25)$$

Writing this expression in $R_n^{(\alpha,\beta)}(f,x)$ and using the identities (6), (7) and (8) we get

$$R_n^{(\alpha,\beta)}(f,x)=f(x)+\frac{f'(x)}{b_n+\beta}\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}\binom{n}{k}(a_n x)^k(k-b_n x)$$

$$+\frac{(\alpha-\beta x)f'(x)}{b_n+\beta}\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}\binom{n}{k}(a_n x)^k$$

$$+\frac{f''(x)}{2(b_n+\beta)^2}\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k(k-b_nx)^2$$
$$+\frac{(\alpha-\beta x)f''(x)}{(b_n+\beta)^2}\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k(k-b_nx)$$
$$+\frac{(\alpha-\beta x)^2f''(x)}{2(b_n+\beta)^2}\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k$$
$$+\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k\left(\frac{k+\alpha}{b_n+\beta}-x\right)^2\lambda\left(\frac{k+\alpha}{b_n+\beta}\right)$$
$$=f(x)+f'(x)\left(-\frac{a_nb_nx^2}{(b_n+\beta)(1+a_nx)}+\frac{(\alpha-\beta x)}{b_n+\beta}\right)$$
$$+\left[\frac{a_n^2b_n^2x^4+b_nx}{2(b_n+\beta)^2(1+a_nx)^2}-\frac{(\alpha-\beta x)a_nb_nx^2}{(b_n+\beta)^2(1+a_nx)}+\frac{(\alpha-\beta x)^2}{2(b_n+\beta)^2}\right]f''(x)+r_n, \tag{26}$$

where

$$r_n=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k\left(\frac{k+\alpha}{b_n+\beta}-x\right)^2\lambda\left(\frac{k+\alpha}{b_n+\beta}\right). \tag{27}$$

Now, for a given arbitrary small number $\varepsilon>0$, there exists $\delta>0$ such that $|t-x|<\delta$ which implies $|\lambda(t)|<\varepsilon$. With such a δ, decompose the sum (27) into two parts:

$$r_n=\sum\nolimits_1+\sum\nolimits_2 \tag{28}$$

where $\sum_1$ contains the members $\left|\frac{k+\alpha}{b_n+\beta}-x\right|<\delta$, and $\sum_2$ the ones $\left|\frac{k+\alpha}{b_n+\beta}-x\right|\geq\delta$. Using the property of $\lambda(t)$ and (8) we obtain

$$\left|\sum\nolimits_1\right|<\varepsilon\left\{\frac{a_n^2b_n^2x^4+b_nx}{(b_n+\beta)^2(1+a_nx)^2}-\frac{2(\alpha-\beta x)a_nb_nx^2}{(b_n+\beta)^2(1+a_nx)}+\frac{(\alpha-\beta x)^2}{(b_n+\beta)^2}\right\}. \tag{29}$$

Now we obtain an upper estimation for $\left|\sum_2\right|$. By $f(t)=O(e^{\gamma t})(t\to\infty,\gamma$ fixed) it follows from (25) for some c_8

$$\left|\lambda\left(\frac{k+\alpha}{b_n+\beta}\right)\left(\frac{k+\alpha}{b_n+\beta}-x\right)^2\right|=\left|f\left(\frac{k+\alpha}{b_n+\beta}\right)-f(x)-f'(x)\left(\frac{k+\alpha}{b_n+\beta}-x\right)-\frac{f''(x)}{2}\left(\frac{k+\alpha}{b_n+\beta}-x\right)^2\right|$$

$$< c_8 e^{\gamma\left(\frac{k+\alpha}{b_n+\beta}\right)} \quad (k = 0,1,2,\ldots,n). \tag{30}$$

By (27), (28), (30) and (9) we get

$$\left|\sum\nolimits_2\right| < \frac{c_9}{(b_n+\beta)^2}\left\{\frac{a_n^2 b_n^2 x^4 + b_n x}{(1+a_n x)^2} + \frac{2(\beta x-\alpha)a_n b_n x^2}{(1+a_n x)} + (\alpha-\beta x)^2\right\}. \tag{31}$$

Let now

$$\rho_n = \frac{r_n}{a_n}. \tag{32}$$

Using (32), (28), (29) and (31) the relation

$$|\rho_n| < \varepsilon\left\{\frac{a_n b_n^2 x^4 + \frac{b_n}{a_n}x}{(b_n+\beta)^2} - \frac{2(\alpha-\beta x)b_n x^2}{(b_n+\beta)^2} + \frac{(\alpha-\beta x)^2}{a_n(b_n+\beta)^2}\right\}$$

$$+\frac{c_9}{(b_n+\beta)^2}\left\{a_n^2 b_n^2 x^4 + b_n x + 2(\beta x-\alpha)a_n b_n x^2 + (\alpha-\beta x)^2\right\} \to 0\ (n\to\infty) \tag{33}$$

holds, because $a_n = \frac{b_n}{n} \to 0$ and $\frac{n^{1/2}}{b_n} \to 0$ as $n\to\infty$. (26), (27), (32) and (33) give the proof of Theorem 4.

Finally, we prove a convergence theorem concerning the derivative of $R_n^{(\alpha,\beta)}(f,x)$.

To prove the theorem we need the following lemma proved in [1].

Lemma 5. In every interval $0 \le x \le A < \infty$, the inequality

$$\frac{1}{(1+a_n x)^n}\left|\sum_{k=0}^{n}(k-b_n x)^m\binom{n}{k}(a_n x)^k\right| \le K_m(A)a_n^m b_n^m \ (m = 0,1,2,\ldots) \tag{34}$$

holds for sufficiently large n, where $K_m(A)$ is a number depending only on A, $a_n = \frac{b_n}{n}$, $b_n = n^{2/3}$.

Theorem 6. Let $f(t)$ be a function defined in $[0,\infty)$ such that $f(t) = O(e^{\gamma t})(t\to\infty, \gamma$ is fixed). If $f'(t)$ exists at the point $t = x$, then

$$\left(R_n^{(\alpha,\beta)}\right)'(f,x) \to f'(x)\ if\ n\to\infty,$$

where $a_n = \frac{b_n}{n} \to 0$ and $b_n = n^{2/3}$.

Proof. Firstly, consider the case $x > 0$. Using (5) and (6) we get

$$\left(R_n^{(\alpha,\beta)}\right)'(f,x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}a_n(a_nx)^{k-1}f\left(\frac{k+\alpha}{b_n+\beta}\right)\left(k-\frac{k+\alpha}{b_n+\beta}\right)$$
$$=\frac{1}{x(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k f\left(\frac{k+\alpha}{b_n+\beta}\right)(k-b_nx)$$
$$=\frac{a_nb_nx}{(1+a_nx)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k f\left(\frac{k+\alpha}{b_n+\beta}\right). \tag{35}$$

Since $f'(x)$ exists and is finite, so

$$f\left(\frac{k+\alpha}{b_n+\beta}\right)=f(x)+\left[f'(x)+\lambda\left(\frac{k+\alpha}{b_n+\beta}\right)\right]\left(\frac{k+\alpha}{b_n+\beta}-x\right), \tag{36}$$

where $\lambda(t)\to 0$ as $t\to x$ .By taking (35), (36), (7) and (8) it follows by simple calculations that

$$\left(R_n^{(\alpha,\beta)}\right)'(f,x)==f'(x)\frac{b_n}{b_n+\beta}\frac{1}{(1+a_nx)^2}+\Delta_n, \tag{37}$$

where

$$\Delta_n=\frac{1}{x(b_n+\beta)}\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k\lambda\left(\frac{k+\alpha}{b_n+\beta}\right)[(k+\alpha)-x(b_n+\beta)]^2$$
$$+\frac{(\beta x-\alpha)}{x(b_n+\beta)}\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k\lambda\left(\frac{k+\alpha}{b_n+\beta}\right)[(k+\alpha)-x(b_n+\beta)]$$
$$+\frac{a_nb_nx}{(b_n+\beta)(1+a_nx)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(a_nx)^k\lambda\left(\frac{k+\alpha}{b_n+\beta}\right)[(k+\alpha)-x(b_n+\beta)]$$
$$=\frac{1}{x(b_n+\beta)}\frac{1}{(1+a_nx)^n}\left\{\sum_{\left|\frac{k+\alpha}{b_n+\beta}-x\right|<\delta}+\sum_{\left|\frac{k+\alpha}{b_n+\beta}-x\right|\geq\delta}\right\}$$
$$+\frac{(\beta x-\alpha)}{x(b_n+\beta)}\frac{1}{(1+a_nx)^n}\left\{\sum_{\left|\frac{k+\alpha}{b_n+\beta}-x\right|<\delta}+\sum_{\left|\frac{k+\alpha}{b_n+\beta}-x\right|\geq\delta}\right\}$$
$$+\frac{a_nb_nx}{(b_n+\beta)(1+a_nx)^{n+1}}\left\{\sum_{\left|\frac{k+\alpha}{b_n+\beta}-x\right|<\delta}+\sum_{\left|\frac{k+\alpha}{b_n+\beta}-x\right|\geq\delta}\right\}$$
$$=A_1+A_2+A_3+A_4+A_5+A_6. \tag{38}$$

Let $\varepsilon > 0$ be an arbitrary number, then by $\lambda(t) \to 0(t \to x)$ there exists a number $\delta > 0$ for which $|\lambda(t)| < \varepsilon$ is valid, if $|t - x| < \delta$, and so by (38) and (8)

$$|A_1| < \varepsilon\left\{\frac{a_n^2 b_n^2 x^3 + b_n}{(b_n + \beta)(1 + +a_n x)^2} - \frac{2(\alpha - \beta x)a_n b_n x}{(b_n + \beta)(1 + a_n x)} + \frac{(\alpha - \beta x)^2}{x(b_n + \beta)}\right\} < c_{10}\varepsilon \quad (39)$$

for sufficiently large n. Similarly,

$$|A_3| < \varepsilon\left\{\frac{(\alpha - \beta x)a_n b_n}{(b_n + \beta)(1 + a_n x)} - \frac{(\alpha - \beta x)^2}{x(b_n + \beta)}\right\} < c_{11}\varepsilon \quad (40)$$

and

$$|A_5| < \varepsilon\left\{\frac{a_n b_n x}{(b_n + \beta)(1 + a_n x)}\left(\frac{-a_n b_n x^2}{1 + a_n x}\right) + \frac{(\alpha - \beta x)a_n b_n x}{(b_n + \beta)(1 + a_n x)}\right\} < c_{12}\varepsilon\,. \quad (41)$$

Since $f(t) = O(e^{\gamma t})(t \to \infty, \gamma$ is fixed), by (36)

$$\left|\lambda\left(\frac{k+\alpha}{b_n+\beta}\right)\right| < c_{13}e^{\gamma\left(\frac{k+\alpha}{b_n+\beta}\right)} \quad if \quad \left|\frac{k+\alpha}{b_n+\beta} - x\right| > \delta\,. \quad (42)$$

We have from (38) and (42)

$$|A_2| \le \frac{c_{13}}{x(b_n+\beta)}\frac{1}{(1+a_n x)^n}\sum_{\left|\frac{k+\alpha}{b_n+\beta}-x\right|>\delta}\binom{n}{k}(a_n x)^k e^{\gamma\left(\frac{k+\alpha}{b_n+\beta}\right)}\left[(k+\alpha) - x(b_n+\beta)\right]^2\,.$$

Applying the Cauchy-Schwarz inequality, we get

$$|A_2| \le \frac{c_{13}}{x(b_n+\beta)}\sqrt{\frac{1}{(1+a_n x)^n}\sum_{\left|\frac{k+\alpha}{b_n+\beta}-x\right|>\delta}\binom{n}{k}(a_n x)^k e^{2\gamma\left(\frac{k+\alpha}{b_n+\beta}\right)}}\,.$$

$$\cdot\sqrt{\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}\binom{n}{k}(a_n x)^k\left[(k - b_n x) + (\alpha - \beta x)\right]^4}\,.$$

Using Lemma 2 as $\xi = 2\gamma$ and Lemma 5, we have

$$|A_2| \le c_{14}\sqrt{\frac{a_n^2 b_n^2 x^4 + b_n x + 2(\beta x - \alpha)a_n b_n x^2 + (\alpha - \beta x)^2}{(b_n+\beta)^4}}\,.$$

$$\cdot\sqrt{K_4(A)a_n^4 b_n^4 + 4(\alpha-\beta x)K_3(A)a_n^3 b_n^3 + 6(\alpha-\beta x)^2 K_2(A)a_n^2 b_n^2 + 4(\alpha-\beta x)^3 K_1(A)a_n b_n + (\alpha-\beta x)^4}$$

$$\to 0\,(n \to \infty). \quad (43)$$

It follows from (36), that

$$\left|\lambda\left(\frac{k+\alpha}{b_n+\beta}\right)\left(\frac{k+\alpha}{b_n+\beta} - x\right)\right| < c_{15}e^{\gamma\left(\frac{k+\alpha}{b_n+\beta}\right)} \quad if \quad \left|\frac{k+\alpha}{b_n+\beta} - x\right| \ge \delta\,. \quad (44)$$

We can obtain $|A_4|$ using (44) and Lemma 2:

$$|A_4| \le \frac{c_{16}(\beta x+\alpha)}{x(b_n+\beta)^2}\left\{\frac{a_n^2 b_n^2 x^4 + b_n x}{(1+a_n x)^2} + \frac{2(\beta x-\alpha)a_n b_n x^2}{(1+a_n x)} + (\alpha-\beta x)^2\right\} \to 0 (n\to\infty). \quad (45)$$

Similarly, we can obtain

$$|A_6| \le c_{17}\left\{\frac{a_n^3 b_n^3 x^5 + a_n b_n^2 x^2 + 2(\beta x-\alpha)a_n^2 b_n^2 x^3 (\alpha-\beta x)^2 a_n b_n x}{(b_n+\beta)^2}\right\} \to 0 (n\to\infty). (46)$$

We can see from (39), (40), (41), (43), (45), (46) that

$$|\Delta_n| \le \sum_{i=1}^{6} |A_i| \le c_{18}\varepsilon$$

for sufficiently large n, thus it follows from (37)

$$\left(R_n^{(\alpha,\beta)}\right)'(f,x) \to f'(x) \; as \; n\to\infty \; and \; x>0.$$

In the other hand, let $x = 0$. Hence

$$\left(R_n^{(\alpha,\beta)}\right)'(f,x)\bigg|_{x=0} = \left\{\frac{1}{(1+a_n x)^n} f\left(\frac{\alpha}{b_n+\beta}\right) + \frac{1}{(1+a_n x)^n} b_n x f\left(\frac{1+\alpha}{b_n+\beta}\right)\right\}'\Bigg|_{x=0}$$

$$= b_n\left[f\left(\frac{1+\alpha}{b_n+\beta}\right) - f\left(\frac{\alpha}{b_n+\beta}\right)\right] \to f'(0),$$

Since $b_n \to \infty$ as $n\to\infty$. This completes the proof of Theorem 6.

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