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THE STRUCTURE OF THE DISCRETE SPECTRUM OF DIFFERENCE OPERATORS OF FIRST ORDER

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ABSTRACT

This paper is a continuation of the paper [8]. In this paper we investigate the dependence of the structure of the discrete spectrum of difference operator of first order on the behavior of coefficients at infinity.

1. INTRODUCTION

The spectral analysis of the selfadjoint difference operators has been investigated in connection with the classical moment problem in [1]. The spectral theory of these operators in selfadjoint case is well known [7]. The spectral analysis of non-selfadjoint difference operators has not been studied extensively. Some of the problems of the spectral theory of non-selfadjoint difference operators have been investigated in [5],[9],[11]. But the spectral analysis of non-selfadjoint differential operators has been studied extensively since 1960's. The spectrum and spectral extension of non-selfadjoint differential operators have been considered by Naimark [12]. He has proved the existence of the spectral singularities on the continuous spectrum of the Strum-Liouville operator. The spectral analyses of the Schrödinger and Klein-Gordon operators with spectral singularities have been studied in [2],[3],[4],[6],[10].

Let L denote the operator generated in $\ell_2(\mathbf{N},\mathbf{C}^2)$ by the difference expression

$$\ell(y)=\begin{pmatrix} a_{n+1}y_{n+1}^{(2)}+b_n y_n^{(2)}+p_n y_n^{(1)} \\ \\ a_{n-1}y_{n-1}^{(1)}+b_n y_n^{(1)}+q_n y_n^{(2)} \end{pmatrix},\ a_0=1,\ n\in N=\{1,2,...\}$$

and the boundary condition

$$y_0^{(1)}=0,$$

where $(a_n),(b_n),(p_n)$ and (q_n) are complex sequences and $a_n\neq 0, b_n\neq 0$ for all $n\in\mathbf{N}$. In this paper we study the dependence of the structure of the discrete spectrum of L on the behaviour of the sequences $(1-a_n),(1+b_n),(p_n)$ and (q_n) at infinity. An analogous problem for the discrete Dirac operator has previously been considered in [5].

2. SPECIAL SOLUTION

Related with the operator L we will consider the equations

$$\begin{aligned} &a_{n+1}y_{n+1}^{(2)}+b_n y_n^{(2)}+p_n y_n^{(1)}=\lambda y_n^{(1)} \\ &a_{n-1}y_{n-1}^{(1)}+b_n y_n^{(1)}+q_n y_n^{(2)}=\lambda y_n^{(2)},\ a_0=1,\ n\in N, \end{aligned} \tag{2.1}$$

Let $(a_n),(b_n),(p_n)$ and (q_n) satisfy the condition

$$\sum_{n=1}^{\infty} n\left(|1-a_n|+|1+b_n|+|p_n|+|q_n|\right)<\infty. \tag{2.2}$$

Under the condition (2.2) it was shown in [8] that if $\lambda=2\sin\frac{z}{2}$ then the system (2.1) has a unique solution which is analytic in the half-plane $\operatorname{Im} z>0$ and continuous upto the real axis having the representation,

$$f_0^{(1)}(z)=\alpha_0^{11}e^{iz/2}\left[1+\sum_{m=1}^{\infty}K_{0m}^{11}e^{imz}\right]-i\alpha_0^{11}\sum_{m=1}^{\infty}K_{0m}^{12}e^{imz} \tag{2.3}$$

and for $n\in N$,

$$f_n(z)=\begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z)\end{pmatrix}=\alpha\left\{E_2+\sum_{m=1}^{\infty}K_{nm}e^{imz}\right\}\begin{pmatrix} e^{iz/2} \\ -i\end{pmatrix}e^{inz} \tag{2.4}$$

where

$$\alpha = \begin{pmatrix} \alpha_n^{11} & 0 \\ \alpha_n^{21} & \alpha_n^{22} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_{nm} = \begin{pmatrix} K_{nm}^{11} & K_{nm}^{12} \\ K_{nm}^{21} & K_{nm}^{22} \end{pmatrix}.$$

Moreover the inequality

$$\left|K_{nm}^{ij}\right| \le C\sigma\left(n + \left[\left|\frac{m}{2}\right|\right]\right), \quad i, j = 1,2, \tag{2.5}$$

holds for all m,n where

$$\sigma(n) = \sum_{k=n}^{\infty}\left(|1 - a_k| + |1 + b_k| + |p_k| + |q_k|\right), \tag{2.6}$$

and $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$ and $C > 0$ is a constant.

3. THE STRUCTURE OF THE DISCRETE SPECTRUM OF L

Let $P_0 = \{z \mid z = \eta + i\tau,\ \tau > 0,\ 0 \le \eta < 4\pi\}$ and $P = P_0 \cup [0,4\pi)$ in complex plane, $\sigma_d(L)$ and $\sigma_{ss}(L)$ denote the discrete spectrum and spectral singularities of L, respectively. It was previously shown that,

$$\sigma_d(L) = \left\{\lambda \mid \lambda = 2\sin\frac{z}{2},\ z \in P_0,\ \beta(z) = 0\right\}$$

$$\sigma_{ss}(L) = \left\{\lambda \mid \lambda = 2\sin\frac{z}{2},\ z \in (0,4\pi) \setminus \{2\pi\},\ \beta(z) = 0\right\}$$

where $\beta(z) = f_0^{(1)}(z)e^{-\frac{iz}{2}}$, [8].

Let us suppose that

$$\sum_{n=1}^{\infty} n^{m+1}\left(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|\right) < \infty \tag{3.1}$$

for $m = 0,1,2,\ldots,k$, and $k < \infty$.

Theorem 3.1. If the condition (3.1) holds, then the function $\beta(z)$ is analytic in P_0 and has continuous derivative of order k on the real axis.

Proof. We obtain from (2.3) that

$$\beta^{(k)}(z) = \alpha_0^{11} i^k \left\{ \sum_{m=1}^{\infty} m^k e^{imz} K_{om}^{11} - i \sum_{m=1}^{\infty} (m - \frac{1}{2})^k e^{i(m-\frac{1}{2})z} K_{om}^{12} \right\}$$

and from (2.5) and (2.6)

$$\left|\beta^{(k)}(z)\right| \le C A_k$$

where

$$A_k = \sum_{n=1}^{\infty} n^k \left(|1 - a_n| + |1 + b_n| + |p_n| + |q_n| \right)$$

for all k. Since (3.1) holds, we have the result.

Using Theorem 3.1 we have following

Corollary 3.1. If the condition (3.1) holds, for all k then the function $\beta(z)$ is infinitely differentiable on the real axis.

Corollary 3.2. If the inequality

$$\sum_{n=1}^{\infty} e^{\varepsilon n^{\gamma}} \left(|1 - a_n| + |1 + b_n| + |p_n| + |q_n| \right) < \infty \tag{3.2}$$

holds where $\varepsilon > 0$ and $0 < \gamma \le 1$, then Corollary 3.1 is valid and also, the inequality

$$\left|\beta^{(k)}(z)\right| \le C B_k$$

holds for all k where

$$B_k = \sum_{m=1}^{\infty} m^k e^{-\varepsilon m^{\gamma}} .$$

Proof. Since the condition (3.2) is stronger than (3.1), the first part is trivial. On the other hand from (2.3), (2.5) and (2.6) we have

$$\left|\beta^{(k)}(z)\right| = \left|\alpha_0^{11}\right| \left| \sum_{m=1}^{\infty} m^k e^{imz} K_{om}^{11} - i \sum_{m=1}^{\infty} (m-1)^k e^{i(m-\frac{1}{2})z} K_{om}^{12} \right|$$

$$\le C_1 \sum_{m=1}^{\infty} m^k e^{-\operatorname{Im} z} \sigma\left(\left[\left| \frac{m}{2} \right| \right] \right)$$

$$\leq 2\,C_1 \sum_{m=1}^{\infty} m^k e^{-\operatorname{Im} z} \sum_{n=m}^{\infty} \left(|1-a_n|+|1+b_n|+|p_n|+|q_n|\right)$$

$$\leq 2\,C_1 \sum_{m=1}^{\infty} m^k e^{-(\operatorname{Im} z+\varepsilon m^{\gamma})} \sum_{n=m}^{\infty} e^{\varepsilon n^{\gamma}} \left(|1-a_n|+|1+b_n|+|p_n|+|q_n|\right)$$

$$\leq C \sum_{m=1}^{\infty} m^k e^{-\varepsilon m^{\gamma}}.$$

Theorem 3.2. If the condition (3.2) holds for $\gamma = 1$ then the function $\beta(z)$ has finite number of zeros with finite multiplicity in P.

Proof. Under the hypothesis, from Corollary 3.2 we have, for all k,

$$\left|\beta^{(k)}(z)\right| \leq C \sum_{m=1}^{\infty} m^k e^{-(\operatorname{Im} z+\varepsilon)m}. \tag{3.3}$$

From (3.3) it is trivial that $\beta(z)$ has analytic continuation to the half-plane $\operatorname{Im} z > -\varepsilon$. So we have the result.

Now let us suppose that

$$\sum_{n=1}^{\infty} e^{\varepsilon n^{\gamma}} \left(|1-a_n|+|1+b_n|+|p_n|+|q_n|\right) < \infty \tag{3.4}$$

where $\varepsilon > 0$ and $\frac{1}{2} \leq \gamma < 1$.

Let us consider the following sets related with the zeros of $\beta(z)$ in the strip P:

$M_1 = \{ z \mid z \in P_0 ,\ \beta(z) = 0 \}$

$M_2 = \{ z \mid z \in [0,4\pi),\ \beta(z) = 0 \}$

$M_3 = \{ z \mid z \in P ,\ \beta^{(k)}(z) = 0,\ k = 0,1,2,\ldots \}$

$M_4 = \{ z \mid \exists \{z_n\} \subset P_0 ,\ \beta(z_n) = 0,\ z_n \to z,\ n \to \infty \}$

$M_5 = \{ z \mid \exists \{z_n\} \subset [0,4\pi) ,\ \beta(z_n) = 0,\ z_n \to z,\ n \to \infty \}$

From the uniqueness theorem of analytic functions it is evident that

$$M_1 \cap M_3 = \varnothing ,\ M_3 \subset M_2 ,\ M_4 \subset M_2 ,\ M_5 \subset M_2$$

and since all derivatives of $\beta(z)$ is continuous up to the real axis we get that

$$M_4 \subset M_3,\ M_5 \subset M_3.$$

On the other hand it can be obtained in a similar way by Lemma 3.4 in [5] that $M_3 = \varnothing$. So $M_4 = \varnothing$ and $M_5 = \varnothing$.

Using above arguments we have the following theorem.

Theorem 3.3. If the condition (3.4) holds then $\sigma_d(L)$ and $\sigma_{ss}(L)$ have finite number of elements with finite multiplicity.

Proof. Let (3.4) hold, then the sets M_1 and M_3 are bounded (see [8], Theorem 3.3 and Theorem 3.4). On the other hand, since M_4 and M_5 , i.e., the set of limit points of M_1 and M_2, respectively, are empty. M_1 and M_2 can have only finite number of elements. Also the elements of M_1 and M_2 have the finite multiplicities since $M_3 = \varnothing$. Therefore the proof is completed.

From Theorem 3.3 it is seen that the weakest condition which guarantees the finiteness of eigenvalues and spectral singularities of L is

$$\sum_{n=1}^{\infty} e^{\varepsilon\sqrt{n}}\left(|1-a_n|+|1+b_n|+|p_n|+|q_n|\right)<\infty\ ,\ \varepsilon>0,$$

which was obtained in [5] for $a_n \equiv 1$ and $b_n \equiv -1$ for all n.

REFERENCES

[1] N.I. Akhiezer, "The Classical Moment Problem and Some Related Questions in Analysis", Ungar, New York, 1965.

[2] E. Bairamov, Ö. Çakar and A.O. Çelebi, Quadratic Pencil of Schrödinger Operators with Spectral Singularities Discretre Spectrum and Principal Functions. J. Math. Anal. Appl. 216, (1997) 303-320.

[3] E. Bairamov, Ö. Çakar and A.M. Krall, Spectrum and Spectral Singularities of a Quadratic Pencil of a Scrödinger Operator with a General Boundary Condition, J. Differential Eq. 151 (1999), 252-267.

[4] E. Bairamov, Ö. Çakar and A.M. Krall, An Eigenfunctions Expansion for a Quadratic Pencil of Schrödinger Operator with Spectral Singularities, J. Differential Eq. 151 (1999), 268-289.

[5] E. Bairamov and A.O. Çelebi, Spectrum and Spectral Expansion for the Non-selfadjoint Discrete Dirac Operators, Quart. J. Math. Oxford (2), 50 (1999), 371-384.

[6] E. Bairamov and A.O. Çelebi, Spectral Properties of the Klein-Gordon s-wave Equation with Complex Potential, Indian J. Pure Appl. Math., 28(6), (1997), 813-824.

[7] Yu.M. Berezanski,"Expansion in Eigenfunctions of Selfadjoint Operators", AMS, Providence, 1968.

[8] C. Coşkun, The Spectrum of the Non-Selfadjoint System of Difference Operators of First Order, Commun. Ank. .Uni. Series A1, Vol. 49 (2000) ... -... .

[9] Yu.L. Kisakevich, On an Inverse Problem for Non-Selfadjoint Difference Operators, Math. Notes, 11, No 6, (1972), 402-407.

[10] V.E. Lyance, A Differential Operator with Spectral Singularities I, II, AMS Translations, (2) 60, (1967), 185-225, 227-283.

[11] F.G. Maksudov, B.P. Allakhverdiev and E. Bairamov, On the Spectral Theory of a Non-Selfadjoint Operator Generated by an Infinite Jacobi Matrix, Soviet Math. Dokl. 43, (1991), 78-82.

[12]M.A. Naimark, Investigation of the Spectrum and the Expansion in Eigenfunctions of a Non-Selfadjoint Operator of the Second Order on a Semi-axis, AMS Translations, (2), 16, (1960), 103-193.