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THE EQUIVALENCE OF 2-GROUPOIDS AND CROSSED MODULES

İ. İÇEN

İnönü University, Science and Art Faculty, Department of Mathematics Malatya-Turkey

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ABSTRACT

The aim of this paper is to give explicit proof of the equivalence of categories 2-groupoids and that of crossed modules, and to explore the relations of coadmissible 2-homotopy and coadmissible homotopy, respectively for 2-groupoids and crossed modules.

1. INTRODUCTION

Higher dimensional category have been considered by various author [4,5,9,11,14]. The category of 2-category contains a category of 2-groupoids [11,14], namely, the 2-category in which all arrows are invertible. This category is equivalent to at least three others categories, of which two are those of crossed modules over groupoids, and of double groupoid with connection [7,15]. Crossed module are more obviously related to classsical tools, namely; groupoids, modules over groupoids, second relative homotopy group and chain complexes [5]. Also the category of crossed module is equivalent to category of G-groupoids (i.e., group object in the category of groupoids) [8,13] and to others mentioned as above.

The structure of the work is as follow: 2-Groupoid and crossed module and their examples are given in Section 1. In Section 2, crossed module associated with a 2-groupoid and 2-groupoid associated with crossed module are explained. In the final section, the relations between the coadmissible 2-homotopies and the coadmissible homotopies are established.

2. Groupoids and Crossed Modules

The material in this section is pretty standard. We choose to use a classical generalization of the notion of a so-called 2-category originally due to Ehresmann [10] and see also Kelly and Street [14]. The 2-categories with invertible arrows (both arrows of dimension 1 and arrows of dimension 2) are so called 2-groupoids. In detail we give the following definition.

Definition 1.1. A **2-groupoid** $\mathbf{H}=\{H_2, H_1, H_0)$ is a set H_2 together with two compatible groupoid structures

$$H_0 = (H, \alpha_0, \beta_0, +_0) \quad H_1 = (H, \alpha_1, \beta_1, +_1)$$

each with H_2 as its set of morphisms. The objects of the groupoid structure H_0, H_1 are regarded as members of H, coinciding with the identity morphisms of H_0, H_1. The compatibility condition are

(I). $\alpha_0 = \alpha_0\alpha_1 = \alpha_0\beta_1$, $\beta_0 = \beta_0\alpha_1 = \beta_0\beta_1$

(II). $\alpha_1(m+_0n) = \alpha_1(m)+_0\alpha_1(n)$ and $\beta_1(m+_0n) = \beta_1(m)+_0\beta_1(n)$, whenever $m,n\in H$ and $m+_0n$ is defined.

(III). (Interchange Law)

$$(m+_0n)+_1(k+_0l) = (m+_1k)+_0(n+_1l)$$

whenever m, n, k, $l\in H$ and both sides are defined.

A 2-groupoid H has objects or 0-cells x etc., arrows or 1-cells $a:x\to y$ etc., and the 2-cells which are often pictured paralel two arrows.

Given a 2-groupoid H, we shall write U(H) for its underlying ordinary groupoid (obtained by leaving out the 2-cells). We can give following examples:

(i). Any ordinary groupoid G can be viewed as a 2-groupoid, with only identity 2-cells.

(ii) Let X be a topological space, let $Y\subseteq X$ be any subspace and let $S\subseteq Y$ be a set of base points. The fundamental groupoid $\pi_1(Y, S)$ on the set S is the underlying groupoid of a 2-groupoid W = W(X, Y, S): the 2-cells in W are homotopy classes of maps from the square $I\times I$ into X, which are constant along the vertical edges with value in S, and the horizontal edges into Y. The domain and codomain of such a deformation are given by restriction to $I\times 0$ and $I\times 1$ respectively. Thus for arrows [a] and [b] from $x\in S$ to $y\in S$ in $\pi_1(Y, S)$, a 2-cell [m] = ([a], [b]) is represented by a mapping $m : I\times I\to X$. Using the homotopy extension property is possible to verify that this gives a well-defined 2-groupoid W; we call it the Whitehead 2-groupoid of (X, Y, S). The related construction of a homotopy double groupoid with connection is treated in full by Brown-Higgins [2].

Definition 1.2. For two 2-groupoids H and K, a homomorphism $\phi : H\to K$ is a function sending the objects, arrows and cells of H to those of K, such that all the structure is preserved.

In particular, the restriction of ϕ to objects and arrows is an ordinary groupoid homomorphism $U(H) \to U(K)$ of underlying groupoids; and for any two objects x and y of H, the restriction of ϕ to arrows $x \to y$ and 2-cells between them yields an ordinary functor $H(x, y) \to K(\phi x, \phi y)$. The 2-groupoids and homomorphisms form a category, written 2-Grpds.

We recall the definition of crossed modules over groupoids. Crossed module were introduced by J.H.C. Whitehead over the groups [16,17]. For the groupoid case, basic references are Brown-Higgins and İ.İçen.

Definition 1.3. Let G, C be groupoids over the same object set and let C be totally intransitive. Then an action of G on C is given by a partially defined function

$$C \times G \to C,$$

written $(c, a) \mapsto c^a$, which satisfies

1. c^a is defined if and only if $\beta(c) = \alpha(a)$, and then $\beta(c^a) = \beta(a)$, where α, β are respectively the source and target maps of the groupoid G.
2. $(c_1 + c_2)^{\alpha} = c_1^{\alpha} + c_2^{\alpha}$,
3. $c_1^{a+b} = (c_1^a)^b$ and $c_1^{e_x} = c_1$ for all $c_1, c_2 \in C(x, x)$, $a \in G(x, y)$, $b \in G(y, z)$.

Definition 1.4. A **crossed module of groupoids** consists of a pair of groupoids C and G over a common object set such that C is totally intransitive, together with an action of G on C, together with a functor $\delta : C \to G$ which is the identity on the object set and satisfies

1. $\delta(c^a) = -a + \delta c + a$
2. $c^{\delta c_1} = -c_1 + c + c_1$

for $c, c_1 \in C(x, x)$, $a \in G(x, y)$.

A crossed module will be denoted by $\mathbf{C} = (C, G, \delta)$. A **crossed module of groups** is a crossed module of groupoids as above in which C, G are groups.

The followings are standard examples of crossed modules:

(i). Let H be a normal subgroup of a group G with $i : H \to G$ the inclusion. The action of G on the right of H by conjugation makes (H, G, i) into a crossed module.

(ii). Suppose G is a group and M is a right G-module; let $0 : M \to G$ be the constant map sending M to the identity element of G. Then (M, G, 0) is a crossed module.

(iii). Suppose given a morphism

$$\eta : M \to N$$

of left G-modules and form the semi-direct product $G \ltimes N$.

This is a group which acts on M via the projection from $G \ltimes N$ to G. We define a morphism

$$\delta : M \to G \ltimes N$$

by $\delta(m) = (1, \eta(m))$ where 1 denotes the identity in G. Then $(M, G \ltimes N, \delta)$ is a crossed module.

Also we can define a category CrsMod of crossed modules of groupoids. Let $\mathbf{C}=(C, G, \delta)$, $\mathbf{C'}=(C', G', \delta)$ be crossed modules. A functor $f=(f_1, f_2) : \mathbf{C} \to C'$ is called a homomorphism of crossed modules, if the maps $f_1 : G \to G'$ and $f_2 : C \to C'$ hold $\delta f_1 = f_2 \delta$, $f_1(c)^{f_2(a)} = f_2(c^a)$.

2.The crossed module associated with a 2-groupoid

Let **H** be a 2-groupoid in the sense of Section 1. Then it has a 2-groupoid structure $\mathbf{H} = (H_2, H_1, H_0)$ satisfying the compatibility conditions (I)-(III).

We shall show that any 2-groupoid **H** contains a crossed module $\mathbf{C} = \lambda\mathbf{H}$, of the kind described in Section 2. We state this as a proposition:

Proposition 2.1. Let $\mathbf{H} = (H_2, H_1, H_0)$ be a 2-groupoid. Then **H** induce a crossed module $\mathbf{C} = (C, G, \delta) = \lambda\mathbf{H}$.

Proof. Given a 2-groupoid **H**, we define $\mathbf{C} = \lambda\mathbf{H}$ by $X = H_0$, $G = H_1$ and

$$C(x) = \{ n \in H_2 \mid \beta_1 n = 1_x \}$$

It follows from (I) that if $n \in C(x)$ $\alpha_0 n = \alpha_0 \ \beta_1 n = x$. Thus we have the alternative characterization:

$$C(x) = \{n \in H \mid \alpha_0 n = \beta_0 n = x.\}$$

Let C be the family $\{C(x)\}_{x \in X}$ and for $n \in C(x)$, define $\delta n = \alpha_1(n)$. Then $\delta n \in G$ since

$$\alpha_0 \ \beta_1 n = \alpha_0 n = x$$

This defines $\delta : C \to G$, $\delta(n) = \alpha_1(n)$ for $n \in C$ and we define

$$\alpha, \beta : G \to X \quad \text{by} \quad \alpha = \alpha_0, \ \beta = \beta_0$$

Clearly G is a groupoid over X with respect to the composition $+_1$. Also for each $x \in X$ and C(x) is a group with respect to each of the compositions $+_i$ for i=0,1, with zero element x. If $m, n \in C(x)$ then, we find that

$$n +_0 m = n +_1 m = m +_0 n$$

Let $n \in C(x)$ and let $a \in G(x, y)$. We define

$$n^a = -a +_0 n +_0 a$$

then

$$\beta_1 n^a = -\beta_1 a +_0 \beta_1 n +_0 \beta_1 a$$

$$= -a +_0 n +_0 a = y$$

and $\alpha_0 c = \alpha_0 a = x$. Thus , in either case, $n^a \in C(y)$ and we obtain an action of G on C. This action is preserved by

$$\delta(n^a) = -\alpha_1 a +_0 \alpha_1 n +_0 \alpha_1 a = -a +_0 \delta n +_0 a.$$

Moreover, if $m \in C(x)$, $u \in H$ and $\alpha_0 u = x$, then

$$-u +_0 m +_0 u = m^{\alpha_1 u}.$$

From (I), we see that if $m, n \in C(x)$, then $-m +_0 n_0 +_0 m = n^{\delta m}$.
This completes the verification that $\mathbf{C} = (C, G, \delta)$ is a crossed module, which we denote by $\lambda\mathbf{H}$. We observe that this crossed module is entirely contained in $\mathbf{H}$, and all its compositions are induced by $+_0$, while its source and target maps are induced by various α_i, β_i. The groups C(x) and C(y) are disjoint if x=y.

Now our aim is to show that H can be recovered from the crossed module $\mathbf{C}$=(C, G, X)= $\lambda\mathbf{H}$ contained in it. We state this as a proposition:

Proposition 2.2. Let $\mathbf{C} = \lambda\mathbf{H}$ be a crossed module over groupoids. Then $\mathbf{C}$ induces a 2-gropoid $K=(K_2, K_1, K_0)$.
Proof. Let $\mathbf{C} = (C, G, \delta)$ be a crossed module over a groupoid. Let K be the set

$$G \ltimes C = \{ (a, c) \mid a \in G,\ c \in C(\beta a)\}$$

We define $\alpha_0(a, c) = \alpha(a)$ and $\beta_0(a, c) = \beta(a)$. So let $K_0 = \alpha_0 K = \beta_0 K$. Suppose now that we are given m=(a, c) , n=(b, c') such that $\beta_0 m = \alpha_0 n$, that is, β $\beta(a) = \alpha(b)$. We define

$$m+_0 n = (a, c)+_0(b, d) = (a+b, c^b+d),$$

which is an element of K.

Similarly, we define $\alpha_1(a, c) = a+\delta c$ and $\beta_1(a, c) = a$.
Let $\beta_1(m) = \alpha_1(n)$. Then we define

$$m+_1 n = (a, c) +_1 (b, c') = (a, c+c'),$$

again an element of K, here $b = a+\delta c$.
Also we have to show that

$$\alpha_1 (m +_0 n) = (\alpha_1 (m) +_0 \alpha_1 (n))$$
$$\beta_1 (m +_0 n) = \beta_1 (m) +_0 \beta_1 (n)).$$

One can easily prove this for α_1. For β_1, let m =(a, c), n= (b, c') , $m+_0 n \in G \ltimes C$, then

$$\begin{aligned}
\beta_1 (m +_0 n) &= \beta_1 ((a, c) +_0 (b, c')) \\
&= \beta_1 (a+b, c^{b+c'}) \\
&= a+b+\delta\ (c^b + c') \\
&= a+b+\delta\ (c^b) + \delta (c') \\
&= a+b-b+\delta (c)+b+\delta (c') \\
&= a+\delta (c)+b+\delta (c') \\
&= \beta_1 (a, c) +_0 \beta_1 (b, c').
\end{aligned}$$

So α_1 and β_1 are morphisms of groupoids.

In each case one can easily see that the composition $+_0$, $+_1$ define groupoid structures on K with K_0, K_1 as their set of identities and α_0, β_0, α_1, β_1 as their source and target maps. The interchange law as follows :

$$((a, c) +_1(a_1, c_1))+_0((b, d)+_1(b_1, d_1)) = (a, c+c_1)+_0(b, d+d_1)$$
$$= (a+b, (c+c_1)^b+d+d_1)$$

and

$$((a, c)+_0(b, d))+_1((a_1, c_1) +_0(b_1, d_1)) =(a+b, c^b+d)+_1(a_1+b_1, c^{b_1} +d_1)$$
$$=(a+b, c^b+d+c^{b_1} +d_1).$$

These are equal if and only if

$$c_1^b + d = d + c_1^{b_1}$$

i.e., $c_1^{b_1} = -d + c_1^b + d = c_1^{b+\delta d}$. On the other hand, for $(b, d) +_1(b_1, d_1)$, we must have

$$b + \delta d = b_1.$$

We can say that the interchange law is exactly equivalent to the 2nd rule for crossed modules.

Now we present the main theorem:

Theorem 2.3. The functors

$$\lambda : \text{2-Grpd} \rightarrow \text{CrsMod}$$

and

$$\theta : \text{CrsMod} \rightarrow \text{2-Grpd}$$

defined above are inverse equivalences.

Proof. Given a 2-groupoid **H**, the 2-groupoid $\boldsymbol{K} = \theta\lambda\mathbf{H}$ is naturally isomorphic to **H** by the

$$(a, c) \rightarrow 1_a +_0 c$$

where $a \in G$, $c \in (\lambda\mathbf{H})$.

The bijection determines the structure on $\boldsymbol{K} = \theta\lambda\mathbf{H}$. This leads us to define 2-groupoid on $\theta(\mathbf{C})$, we shall recall that $\alpha_1(a, c) = a + \delta c$, $\beta_1(a, c) = a$. The map $\theta\lambda$ preserves $+_0$ and $+_1$;

$$\theta\lambda((a, c) +_0 (b, d)) = \theta\lambda(a+b, c^b+d)$$
$$= \theta\lambda(a+b, -1_b+c+1_b+d)$$
$$= 1_{a+b}-1_b+_0c+_0 1_b+_0 d$$
$$= 1_a+_0 c+_0 1_b +_0 d$$
$$=\theta\lambda(a, c) +_0 \theta\lambda(b, d)$$

On the other hand, if **C** is a crossed module, $\mathbf{H} = \theta(\mathbf{C})$ and $D = \lambda\theta(\mathbf{C})$, then **C** consists of element $m = (a, c)$ and so

$$\boldsymbol{D} = \{m \in C \mid \beta_1(m) = 1_x\}$$

consists of elements $m = (1_x, c)$, where $c \in C(x)$. It is easy to see that the map $\mathbf{C} \rightarrow \boldsymbol{D}$ defined by

$$c \mapsto (1_x, c), \quad c \in C(x)$$

gives a natural isomorphism $C \to \lambda\,\theta\, C$. The morphism $\lambda\,\theta$ preserves the structures. Indeed, $\lambda\,\theta\,(c +_0 d) = (1_x\, c+d) = (1_x + 1_x, c^{1_x} + d) = (1_x, c) + (1_x, d) =$ $\lambda\,\theta\,(c) +_0 \lambda\,\theta\,(d)$ and $\lambda\,\theta\,(c +_1 d) = (1_x, c+d) = \lambda\,\theta\,(c) +_1 \lambda\,\theta\,(d)$.

3. Homotopies of crossed modules and 2-groupoids

The notion of homotopy for morphisms of crossed modules over groups has been well known for many years [16],[17]. This was put in the general context of a monodial closed structure on the category of crossed complexes in [5]. The homotopy of crossed module over groupoid has been explored by the author [12].

In this section, we explain the relation between homotopies for crossed modules over groupoids and homotopies for 2-groupoids. The formulae given below are playing important role in our study.

Definition 3.1. Let $\mathbf{C} = (C, G, \delta)$ be a crossed module over groupoids with base space X. A **coadmissible homotopy** s is a pair of maps $s_0 : X \to G$, $s_1 : G \to C$ which satisfy the following

a) $\beta(s_0 x) = x$, $x \in X$ and $\beta(s_1 a) = \beta(a)$, $a \in G$,

b) $s_1(a + b) = s_1(a)^b + s_1(b)$, $a, b \in G$.

c) $f = (f_0, f_1, f_2)$, defined as follows,

$$f_0(x) = \alpha\, s_0(x),$$
$$f_1(a) = s_0(\alpha\, a) + a + \delta\, s_1(a) - s_0(\beta\, a),$$
$$f_2(c) = (c + s_1\,\delta\, c)^{-s_0\beta(c)}$$

is an automorphism $f = (f_0, f_1, f_2)$ of $\mathbf{C}$.

The notion of homotopy for 2-groupoids is essentially a special case of 2-natural transformation due to Gray in [11].

Definition 3.2. [6] Let $\mathbf{H} = (H_2, H_1, H_0)$ be a 2-groupoid and let f be an automorphism of *H*. A pair (σ_0, σ_1) where $\sigma_0 : H_0 \to H_1$ and $\sigma_1 : H_1 \to H_2$, is called a **coadmissible 2-homotopy** of f if

(a). $\alpha\,\sigma_0(x) = f(x)$ and $\beta\,\sigma_0(x) = x$,

(b). If $a \in H_1(x, y)$,

$$\alpha\,\sigma_1(a) = -\sigma_0(x) + f(a) + \sigma_0(y)$$
$$\beta\,\sigma_1(a) = a \text{ and } \beta_0\sigma_1(a) = \beta\,(a).$$

(c). $\sigma_1(a+b) = \sigma_1(a) +_0 \sigma_1(b)$ whenever a+b is defined in H_1.

(d). for each $m \in H$ with $\alpha_1(m) = a$, $\beta_1(m) = b$, we have

$$m +_1 \sigma_1(b) = \sigma_1(a) +_1 f(m).$$

Proposition 3.3. Let $\mathbf{H} = (H_2, H_1, H_0)$ be a 2-groupoid and let f, g be automorphisms of **H** and, let $\sigma : g \cong I$, $\tau : f \cong I$ be homotopies. Then we can define a coadmissible 2-homotopy

$$\sigma * \tau : gf \cong I_H$$

by

$$(\sigma * \tau)_0(x) = \tau_0(f(x)) + \sigma_0(x)$$

and

$$(\sigma * \tau)_1(a) = \sigma_1(a) +_1 \tau_1(f(a))$$

for each $x \in X$, $a \in G$.

Proof. We verify the condition a), b), c) of Definition 3.2 for $\sigma * \tau$,.i.e.,

a) $\beta(\sigma * \tau)_0(x) = x$,
$\alpha(\sigma * \tau)_0(x) = gf(x)$, $x \in X$.

b) If $a \in G(x, y)$,
$\alpha(\sigma * \tau)_1(a) = -(\sigma * \tau)_0(x) + gf(a) + (\sigma * \tau)_0(y)$. $\beta(\sigma * \tau)_1(a) = a$,
$\beta_0(\sigma * \tau)_1(a) = \beta(a)$,
and $(\sigma * \tau)_1$ is a linear map,i.e.,
$(\sigma * \tau)_1(a+b) = (\sigma * \tau)_1(a) +_0 (\sigma * \tau)_1$

c) For any $m = (a, b) \in H_2$,
$m +_1 (\sigma * \tau)_1(b) = (\sigma * \tau)_1(a) +_1 gf(m)$.

In fact,

$$\begin{aligned}\alpha(\sigma * \tau)_0(x) &= \alpha(\sigma_0(f(x)) +_0 \tau_0(x)\\ &= \alpha(\sigma_0(f(x))\\ &= \alpha(\sigma_0) f(x)\\ &= g f(x)\end{aligned}$$

and also similarly we obtain

$$\begin{aligned}\beta(\sigma * \tau)_0(x) &= \beta(\sigma_0(f(x)) + \tau_0(x)\\ &= \beta \tau_0(x)\\ &= x.\end{aligned}$$

For linearity, suppose $a, b \in H_1$ and a+b is defined. Then

$$\begin{aligned}(\sigma * \tau)_1(a+b) &= \sigma_1(a+b) +_1 \tau f(a+b) \text{ by definition*}\\ &= (\sigma_1(a) +_0 \sigma_1(b)) +_1 \tau(f(a) +_0 \tau f(b)), \text{ by linearity}\\ &= (\sigma_1(a) +_1 \tau f(a)) +_0 (\sigma_1(b) +_1 \tau f(b)), \text{ Interchange Law}\\ &= (\sigma_1 * \tau_1)(a) +_0 (\sigma_1 * \tau_1)(b).\end{aligned}$$

Proposition 3.4. Let $\mathbf{H} = (H_2, H_1, H_0)$ be a 2-groupoid. Then M(**H**), the set of coadmissible 2-homotopies of **H**, is a group with respect to the multiplication * given in Proposition 3.3, with identity constant homotopy $c : I \cong I$ and inversion $\sigma^{-1} : f^1 \cong I$ for a coadmissible 2-homotopy $\sigma : f \cong I$.

Proof. It has been proved that in previous Proposition 3.3 that if σ, τ are two coadmissible 2-homotopies of **H**, then $\sigma * \tau$ is a coadmissible 2-homotopy of **H**. Also one can easily show that the constant 2-homotopy $c : I \cong I$ is a constant map. In order to define an inverse element let $\sigma : f \cong I$ be a coadmissible 2-homotopy. An inverse coadmissible 2-homotopy is given by $\sigma^{-1} : f^{1} \cong I$; where $\sigma_0^{-1} = -\sigma_0(f^{-1}(x))$ and $\sigma_1^{-1}(a) = -\sigma_1(f^{-1}(a))$. One can easily show that $\sigma * \sigma^{-1}=c$ and $\sigma^{-1} * \sigma = c$, as follows:

$$\begin{aligned}(\sigma^{-1} * \sigma)_1(a) &= \sigma^{-1}(a) + \sigma(f^{1}(a)) \\ &= -\sigma f^{1}(a) + \sigma(f^{1}(a)) \\ &= c(a).\end{aligned}$$

and

$$\begin{aligned}(\sigma * \sigma_1^{-1})_1(a) &= \sigma_1(a) + \sigma_1^{-1}(f(a)) \\ &= \sigma_1(a) - \sigma_1(f^{1}(f(a))) \\ &= \sigma_1(a) - \sigma_1(a) \\ &= c(a).\end{aligned}$$

Theorem 3.5. Let $\lambda : \text{2-Grpd} \to \text{CrsMod}$ be the natural transformation as defined in Theorem 2.1 and let $\sigma : f \cong I : \mathbf{H} \to \mathbf{H}$ be a coadmissible 2-homotopy of $\mathbf{H}$. Then $\lambda(\sigma) : \lambda(f) \cong \lambda(I)$ is a coadmissible homotopy for corresponding crossed module $\mathbf{C}=(G \ltimes C, G, \delta)$. If further, then $\lambda(\sigma * \tau) = \lambda(\sigma) * \lambda(\tau)$.

Proof. Let $m = (a, c) \in H_2$ and $a, c \in H_1(x, y)$. By definition of λ, $\lambda(a,c)= (I_y, \alpha(m))$. If $\sigma_1(a) = (a, -\sigma_0 + f(a) + \sigma_0(y)$, then clearly $\lambda\sigma_1(a) = (1_y, -a-\sigma_0 + f(a) + \sigma_0(y))$. In fact, by definition δ in crossed module $\mathbf{C}$, $\alpha_1(\lambda\sigma_1(a)) = \delta(\lambda\sigma_1(a))$. If we write $\lambda\sigma_1 = s_1$, then we obtain

$$\delta s_1(a) = -a - \sigma_0(x) + f(a) + \sigma_0(y).$$

Moreover, if $a, b \in H_1$ and $a+b$ is well-defined, then

$$\lambda\sigma_1(a+b) = \lambda\sigma_1(a)^b + \lambda\sigma_1(b),$$

i.e., $\lambda\sigma_1$ is a derivation map.

$$\lambda\sigma_1(a+b) = -(a+b) - \sigma_0(x) + f(a+b) + \sigma_0(z) \text{....(I)}$$

and $\lambda\sigma_1(a)^b = -a - \sigma_1(a) = (a, \sigma_0 + f(a) + \sigma_0(y))$, $\lambda\sigma_1(b) = -b - \sigma_0(y) + f(a) + \sigma_0(z)$. Then $\lambda\sigma_1(a)^b + \lambda\sigma_1(b) = -b-a-\sigma_0 + f(a) + \sigma_0(y) + b - b - \sigma_0(y) + f(a) + \sigma_0(z) = (a+b) - \sigma_0(x) + f(a) + f(b) + \sigma_0(z)$... (II).

Since (I) and (II), $\lambda\sigma_1$ is a derivation map. Hence

$$\lambda(\sigma) : \lambda(f) \cong \lambda(I)$$

Moreover, if $\tau : g \cong 1$, then $\lambda(\sigma * \tau) = \lambda(\sigma\lambda(\tau)$. In fact, $\lambda(\sigma * \tau)_1(a) = \lambda\sigma_1(a) +_0 (\lambda\tau_1 f(a))^{\sigma_0(y)}$. $\lambda\sigma_1(a) = -a\sigma_0(x) - \tau_0 f(x) + gf(a) + \tau_0 f(y) + \sigma_0(y)$, $\lambda\sigma_1(a) = -a - \sigma_0(x) + f(a) + \sigma_0(y)$, and $\lambda\tau_1 f(a) = -f(a) - \tau_0 f(x) + gf(a) + \tau_0 f(y)$. Thus $\lambda\sigma_1(a) + \lambda\tau_1 f(a) = \lambda\sigma_1(a) - \sigma_1(y) + \lambda\tau f(a) + \sigma_0(y) = \lambda\sigma_1(a) + (\lambda\tau f(a))^{\sigma_0(y)}$.

i.e., $\lambda(\sigma * \tau) = \lambda(\sigma) * \lambda(\tau)$.

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