Commun. Fac. Sci. Univ. Ank. Series A1
V. 49. pp. 33 - 37 (2000)

CONTACT RIEMANNIAN MANIFOLDS SATISFYING C(ξ,X)S=0 AND ξ∈(k, μ)-NULLITY DISTRIBUTION

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(Received Jan 12, 1999; Revised April 09, 2000 ; Accepted April 18, 2000)

SUMMARY

This study deals with a classification of the contact manifolds satisfying $C(\xi,X)S=0$, where S, C are the Ricci tensor and Weyl tensor, respectively, under the conditon that characteristic vector field ξ belongs to the (k, μ)-nullity distribution.
In this

1.INTRODUCTION

Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a contact Riemannian manifold of dimension $2n+1>3$. Tanno [6] proved that $(M^{2n+1},\varphi,\xi,\eta,g)$ is an Einstein manifold and ξ belongs to the k-nullity distribution, then M is a Sasakian manifold and Perrone [4] proved that if M is a contact Riemannian manifold with $R(X,\xi)S=0$ and ξ belongs to the k-nullity distribution, where $k\in \mathbf{R}$, then M is either an Einstein-Sasakian manifold or the product $E^{n+1}(0)\times S^n(4)$. Papantoniou [1] generalizing this result proved that if M is a contact Riemannian manifold with $R(X,\xi)S=0$ and ξ belongs to the (k,μ)-nullity distribution, where $(k,\mu)\in \mathbf{R}^2$, then M is local isometric to $E^{n+1}(0)\times S^n(4)$ or an Einstein-Sasakian manifold or, an η-Einstein manifold.The purpose of this paper is to classify the contact manifolds satisfying $C(X,\xi)S=0$ under the condition that characteristic vector field ξ belongs to the (k,μ)-nullity distribution.

2.PRELIMINARIES AND KNOWN RESULTS

Let M^{2n+1} be an (2n+1) dimensional differentiable manifold. If there exist in M^{2n+1} a global differential 1-form η satisfying $\eta\wedge(d\eta)\neq 0$ then M^{2n+1} is called contact manifold. It is well known that a contact manifold admits a vector field ξ, called the characteristic vector field, such that $\eta(\xi)=1$ and $d\eta(\xi,X)=0$ for every $X\in\chi(M)$.
Moreover, M admits a Riemannian metric g and tensor field φ of type (1.1) such that

$$\varphi^2= -Id+\eta\otimes\xi,\ \ g(X, \xi)=\eta(X),\ \ g(X,\varphi Y)=d\eta(X,Y) \tag{2.1}$$

then (φ,ξ,η,g) is called a contact metric structure. If M^{2n+1} has a contact metric structure then, $(M^{2n+1}, \varphi,\xi,\eta,g)$ is called contact metric manifold.

As a consequence of the relation (2.1) we have

$$g(\varphi X,\varphi Y)= g(X,Y)- \eta(X)\eta(Y),\ \varphi\xi=0,\ \eta\varphi=0. \tag{2.2}$$

Denoting by L and R, Lie differentitation and curvature tensor, respectively, we define the operators ℓ and h by

$$\ell X=R(X,\xi)\xi,\ hX=\frac{1}{2}(L_\xi\varphi)X. \tag{2.3}$$

The (1.1) tensors h and l self-adjoint and satisfy

$$h\xi=0,\ l\xi=0,\ Trh=Trh\varphi=0,\ h\varphi=-\varphi h. \tag{2.4}$$

If ∇ is the Riemannian connection of (M,g), then

(2.5) i) $\nabla_X\xi=-\varphi X-\varphi hX$,

ii) $\nabla_\xi\varphi=0$,

iii) $\varphi l\varphi-l=2(h^2+\varphi^2)$.

See also [3].

A contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y.$$

On the manifold M endomorphisms $X\wedge Y$ and $C(X,Y)$ are defined by

$$(X\wedge Y)Z=g(Y,Z)X-g(X,Z)Y \tag{2.6}$$

$$C(X,Y)Z=R(X,Y)Z-\frac{-1}{(n-2)}[(X\wedge Y)Z+(QX\wedge Y)Z]+\frac{1}{(n-2)(n-1)}\kappa(X\wedge Y)Z \tag{2.7}$$

respectively, where the Ricci operator Q of (M,g) is defined by

$$S(X,Y)=g(QX,Y) \text{ and } X,Y,Z\in\chi(M) \tag{*}$$

and $n>2$.

We extend these endomorphisms to the derivations $C(X,Y)$ and $X\wedge Y$ of the algebra of the tensor fields on M, assuming that they commute with contractions and

$$C(X,Y)f=0,\ (X\wedge Y)f=0$$

for every smooth function on M where, C(X,Y) is the Weyl tensor of M.

A contact metric manifold is said to be η-Einstein if

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),$$

where a, b are smooth functions on M.

The (k,μ)-nullity distribution of a contact metric manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ for the pair $(k, \mu)\in\mathbf{R}^2$, is a distribution given by;

$N(k, \mu):p\to N_p(k, \mu)=\{R(X,Y)Z\} =\ k[g(Y,Z)X-g(X,Z)]+ \mu[g(Y,Z)hX- g(X,Z)hY]$

So, if the characteristic vector field ξ belongs to the (k,μ)-nullity distribution we have

$$R(X,Y)\xi=k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY]. \qquad (2.8)$$

3.CONTACT RIEMANNIAN MANIFOLDS WITH C(ξ,X)S=0 AND ξ∈(K,μ)-NULLITY DISTRIBUTION

In this section we will use the following well known results to prove the main theorem. We will use next the following Theorem and Lemma.

Theorem 3.1[3]: Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a contact metric manifold with $R(X,Y)\xi=0$ for all vector fields X,Y. Then M^{2n+1} is locally product of flat (n+1)-dimensional manifold and an n-dimensional manifold of positive contact curvature equal to 4.

Lemma 3.2[2]: In any contact metric manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ with ξ belonging to the (k,μ)-nullity distribution, where k<1, the Ricci operator Φ is given by

$$QX=[2(n-1)-n\mu]X+[2(n-1)+\mu]hX+[2(1-n)+n(2k+\mu]\eta(X)\xi,\ n\geq 1 \qquad (3.1)$$

for any $X\in\chi(M)$.

Corollary 3.3[2]: In any contact metric manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ with ξ belonging to the (k,μ)-nullity distribution the Ricci tensor S is given by

$$S(X,Y)=[2(n-1)-n\mu]g(X,Y)+[2(n-1)+\mu]g(hX,Y)$$
$$+[2(1-n)+n(2k+\mu]\eta(X)\eta(Y),\ (n\geq 1). \qquad (3.2)$$

Main Theorem: Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a contact metric manifold given with conditions

i)$C(\xi,X)S=0$,

ii) $R(X,Y)\xi=k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY].,\forall(k,\mu)\in\mathbf{R}^2$ (k<1)

Then the M^{2n+1} is either

i)locally isometric to $E^{n+1}(0)\times S^n(4)$ or,

ii)an η-Einstein manifold.

Proof: Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a contact metric manifold which is the given conditions . We have the following cases;

Case (i): If k=0, μ=0 then we have $R(X,Y)\xi=0$ for all $X,Y\in\chi(M)$ and so by Theorem 3.1, the manifold M is locally isometric to $E^{n+1}\times S^n(4)$.

Case (ii): If k≠0 and μ≠0, then from the first hypothesis we have

$$0=(C(\xi,X)S)(Y,Z)=C(\xi,X)S(Y,Z)-S(C(\xi,X)Y,Z)-S(Y,C(\xi,X)Z) \qquad (3.3)$$

and from which

$$S(C(\xi,X)Y,Z)=-S(Y,C(\xi,X)Z)\quad \forall X,Y,Z\in\chi(M). \qquad (3.4)$$

If we take $Z=\xi$ in relation (3.4), we get

$$S(C(\xi,X)Y,\xi)= -S(Y,C(\xi,X)\xi), \tag{3.5}$$

and so we get

$$S(C(\xi,X)Y,\xi)= 2nkg(C(\xi,X)Y,\xi). \tag{3.6}$$

Using equation (3.1) we have

$$C(\xi,X)\xi=R(\xi,X)\xi- \frac{2nk}{(n-2)}\eta(X)\,\xi + \frac{1}{(n-2)}QX+\frac{2nk}{(n-2)}X + \frac{1}{(n-2)(n-1)}\kappa(\eta(X)\xi-X) \tag{3.7}$$

and using the second hypothesis, we also get

$$S(C(\xi,X)\,\xi,Y)=\frac{1}{(n-2)}S^2(X,Y)+(A_1-k)S(X,Y) +(B_1+2nk^2)\,\eta(X)\eta(Y)-\mu S(hX,Y). \tag{3.8}$$

where

$$S^2(X,Y)=g(QX,QY) \tag{3.8*}$$

and

$$A_1=\frac{2nk}{(n-2)}-\frac{1}{(n-2)(n-1)}\kappa,\quad B_1=\frac{2\kappa nk}{(n-1)(n-2)}-\frac{4n^2k^2}{(n-2)}.$$

Using the equations (3.7), (3.3) and the second hypothesis we get

$$-S(C(\xi,X)Y,\xi)=\frac{2nk}{(n-2)}S(X,Y)-A_2g(X,Y)-B_2\eta(X)\eta(Y)-2nk\mu g(hX,Y) \tag{3.9}$$

where,

$$A_2=2nk^2+\frac{4n^2k^2}{(n-2)}+\frac{2\kappa nk}{(n-1)(n-2)},\quad B_2=-2nk^2-\frac{2n\kappa}{(n-1)(n-2)}.$$

Using (3.8) and (3,9), the equation (3,4) yields to

$$\frac{1}{(n-2)}S^2(X,Y)+(A_1-k)S(X,Y)+(B_1+2nk^2)\,\eta(X)\eta(Y)-\mu S(hX,Y) \tag{3.10}$$

$$=\frac{2nk}{(n-2)}S(X,Y)-A_2g(X,Y)-B_2\eta(X)\eta(Y)-2nk\mu g(hX,Y).$$

Now let us calculate $S^2(X,Y)$; since the tensor h is self-adjoint and $h^2=(k-1)\varphi^2$, $k\leq 1$, (see[2]) we have

$$S^2(X,Y)=(A^2+(k-1)B^2)g(X,Y)+(2AC+C^2-(k-1)B^2)\,\eta(X)\eta(Y) \tag{3.11}$$

where, $A=2(n-1)-n\mu$, $B=2(n-1)+\mu$, and $C=2(1-n)+n(2k+\mu)$.

Furthermore, using the equation (3.2) we obtain

$$S(hX,Y)=Ag(hX,Y)-(k-1)Bg(X,Y)+(k-1)\,\eta(X)\eta(Y). \tag{3.12}$$

Combining the equations (3.11) and (3.12) with (3.10), we get

$$A_3S(X,Y)+\mu B_3g(hX,Y)=A_4g(X,Y)+B_4\eta(X)\eta(Y), \tag{3.13}$$

where,

$$A_3=A_1-k-\frac{2nk}{(n-2)},\ B_3=2nk-A,\ A_4=-[\mu(k-1)B+\frac{1}{(n-2)}(A^2+(k-1)B^2)-A_2]$$

$$B_4=-[\frac{1}{(n-2)}(2AC+C^2-(k-1)B^2)+B_1+2nk^2-\mu(k-1)B+B_2].$$

By (3.12) and (3.13) we also get

$$g(hX,Y)=(\frac{A_4-A_3A}{A_3B+\mu})g(X,Y)+(\frac{B_4-A_3C}{A_3B+\mu})\ \eta(X)\eta(Y). \qquad (3,14)$$

Hence by (3.13) and (3.14) we obtain

$$S(X,Y)=(\frac{A_4-\mu B_3A_5}{A_3})g(X,Y)+(\frac{A_4-\mu B_3B_5}{A_3})\ \eta(X)\eta(Y), \qquad (3,15)$$

where,

$$A_5=\frac{A_4-A_3A}{A_3B+\mu},\quad B_5=\frac{B_4-A_3C}{A_3B+\mu}. \qquad (3,16)$$

So $(M^{2n+1},\varphi,\xi,\eta,g)$ is an η-Einstein manifold.
If $k\neq0$ and $\mu=0$, then it is easy to show that $(M^{2n+1},\varphi,\xi,\eta,g)$ is also an η-Einstein manifold and this completes the proof of theorem.

REFERENCES

[1] B.J.Papantoiou, Contact Riemannian manifolds R(ξ,X)R=0 and ξ∈(k,μ)-nullity distribution, Yokohama Mathematical Journal, 40, (1993), 149-161.
[2] D.E.Blair, T.Koufogiorgos and B.J.Papantoiou, Contact metric manifolds satisfying a nullity condition, Israel Journal of Mathematics 91,(1995), 189-214.
[3] D.E.Blair, Contact Manifolds in Riemannian Geometry, Lectures Notes in Mathematics 509, Springer-Verlag, Berlin, 1976.
[4] D.Perrone, Contact Riemannian manifolds satisfying R(X,ξ)R=0, (submitted).
[5] S.Tanno, A class of Riemannian manifolds R(X,Y)R=0, Nagoya Math.J., 42, (1971), 67-77.
[6] ______, Ricci curvatures of contact Riemannian manifolds, Tohoku Math. J.,40, (1983), 441-448.