

NUMERICAL SCHEME FOR A COMPLETE ABSTRACT SECOND ORDER DIFFERENTIAL EQUATION OF ELLIPTIC TYPE

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ABSTRACT

We give new results of stability and convergence of a numerical scheme for a complete abstract second order differential equation of elliptic type by using finite difference method. First, the theory of linear operators sums is referred to. Next, we deal with the previous problems using the explicit resolution of an infinite linear system.

KEYWORDS

Numerical scheme, finite differences, abstract differential equations, linear operator sums, a priori estimate, convergence, infinite matrix, Banach algebra, solution of an infinite linear system.

AMS SUBJECT CLASSIFICATION : 35J25, 40C05, 46A45, 65M06

1. INTRODUCTION

In this work, we study the following abstract second order differential equation

$$u''(t) + 2Bu'(t) + Au(t) = f(t), \quad t \in (0,1) \quad (1)$$

under the nonhomogenous boundary conditions

$$\begin{cases} u(0) = \varphi \\ u(1) = \psi \end{cases} \quad (2)$$

where φ, ψ and $f(t)$ belongs to a complex Banach space E . A and B are two closed linear operators with domains $D(A)$ and $D(B)$. Our aims is to study the stability and the convergence of a numerical scheme using finite differences ([6], [8]).

Let N be an enough great positive natural number, putting : $h = \frac{1}{N+1}$, we consider the following problems:

$$\left\{ \begin{array}{l} \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + 2B \left(\frac{v_{j+1} - v_j}{h} \right) + Av_j = f_j \\ v_0 = \varphi \quad v_{N+1} = \psi \end{array} \right. \quad 1 \leq j \leq N \quad (3)$$

where v_j is the approximative value of $u(jh)$, $f_j = f(jh)$, and:

$$\left\{ \begin{array}{l} \frac{w_{j+1} - 2w_j + w_{j-1}}{h^2} + 2B \left(\frac{w_{j+1} - w_j}{h} \right) + Aw_j = f_j \\ w_0 = 0 \quad w_{N+1} = 0 \end{array} \right. \quad 1 \leq j \leq N \quad (4)$$

We assume the next hypothesis:

(H 1) B is the generator of a strongly continuous group in E .

(H2) $\exists K > 0 / \forall \lambda \geq 0 \left\| (A - B^2 - \lambda I)^{-1} \right\|_{L(E)} \leq \frac{K}{1+\lambda}$.

(H 3) $\left\{ \begin{array}{l} (A - B^2 - \lambda I)^{-1} (B - \mu I)^{-1} - (B - \mu I)^{-1} (A - B^2 - \lambda I)^{-1} = 0 \\ \forall \mu \in \mathbb{R} \quad \forall \lambda \geq 0 \end{array} \right.$

(H 4) $\left\{ \begin{array}{l} \exists K > 0 / \forall \lambda \geq 0 \\ i) \left\| B(A - B^2 - \lambda I)^{-1} \right\|_{L(E)} \leq \frac{K}{1+\sqrt{\lambda}} \\ ii) \left\| B^2(A - B^2 - \lambda I)^{-1} \right\|_{L(E)} \leq K. \\ iii) \left\| A(A - B^2 - \lambda I)^{-1} \right\|_{L(E)} \leq K. \end{array} \right.$

(H1) implies that $\overline{D(B)} = E$; but $D(A - B^2)$ is not necessarily dense. The hypothesis (H2), (H3) and (H4) allows us to apply the theory of linear operators sum's Da Prato-Grisvard [1]. The hypothesis (H2) and (H4) express the ellipticity of (1), (2) (see [4]). In Labbas-El Haial [5], the authors give a result of existence, uniqueness and maximal regularity of the strict solution u when f is holderian, where: $u \in C([0,1]; E)$ is a strict solution of (1), (2) if and only if : u verifies (1)-(2) and

$$u \in C^2([0,1]; E) \cap C^1([0,1]; D(B^2)) \cap C([0,1]; D(A)).$$

Our aims is to find an a priori estimate that is fundamental for the study of the stability and the convergence of the numerical scheme.

This work is structured as follows, in the second section, we use the linear operators sum's theory. This method is based in the built of the natural representation of the eventual solution by using operational calculus and the Dunford's integral (see [1]) in some interpolation spaces characterized in [3]. The result obtained generalize the one given in Medeghri [8] (case where $B=0$). In the third section, we recall definitions and properties of the sequence spaces S_1 and l^∞ [6], [7] and [2]. S_1 being a unital Banach algebra, we obtain existence and

uniqueness of the solution of an infinite linear system. Then we get an approximation method for boundary problems of a second order differential equation, using the numerical scheme. Convergence and stability are obtained by the explicit resolution of an infinite linear system. These results generalize in a certain sense those given in [6].

2. APPLICATION OF LINEAR OPERATORS SUM'S THEORY

2.1. Construction of the solution

Writing the problem (3) in the form

$$\Gamma_h V + \Theta_h \beta V + \tilde{A} V = F \quad (I)$$

in $\hat{E} = E^N$, with

$${}^t V = (v_1, v_2, \dots, v_N); \quad {}^t F = (f_1, f_2, \dots, f_N)$$

and

$$\|V\|_{\hat{E}} = \max_{1 \leq j \leq N} \|v_j\|_E.$$

Γ_h is the following N order tridiagonal matrix

$$\Gamma_h = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \cdot & \cdot & 0 \\ 1 & -2 & 1 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 1 & -2 & 1 \\ 0 & \cdot & \cdot & 0 & 1 & -2 \end{pmatrix}$$

Θ_h is the following N order bidiagonal matrix

$$\Theta_h = \frac{2}{h} \begin{pmatrix} -1 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & -1 & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & -1 \end{pmatrix}$$

and

$$\begin{cases} {}^t(\beta V) = (Bv_1, Bv_2, \dots, Bv_N) \\ \text{for } V \in D(\beta) = [D(B)]^N \end{cases}$$

$$\begin{cases} {}^t(\tilde{A}V) = (Av_1, Av_2, \dots, Av_N) \\ \text{for } V \in D(\tilde{A}) = [D(A)]^N. \end{cases}$$

The solution of problem (I) is given by

$$V = \frac{-1}{2i\pi} \int_{\gamma} (A - B^2 - \lambda I)^{-1} Y d\lambda$$

with γ a sectorial boundary curve of the following sector of the complex plane

$$S(\delta_0, r_0) = \{z \in \mathbb{C} : |\arg z| \leq \delta_0\} \cup B(0, r_0),$$

oriented positively and $B(0, r_0)$ is the open ball of radius r_0 . The hypothesis (H2) implies the existence of $\delta_0 \in]0, \pi/2[$ and $r_0 > 0$ such that the resolvent set of $A - B^2$ contains $S(\delta_0, r_0)$.

$${}^tY = (y_1, y_2, \dots, y_N)$$

where

$$y_j = \frac{\sinh \alpha(N+1-j)}{\sinh \alpha(N+1)} e^{-jhB} \varphi + \frac{\sinh \alpha j}{\sinh \alpha(N+1)} e^{(N+1-j)hB} \psi + \sum_{k=0}^{N+1} K_{jk}^\alpha e^{-B(j-k)h} f_k$$

and

$$K_{jk}^\alpha = -h^2 \begin{cases} \frac{\sinh \alpha k \cdot \sinh \alpha(N+1-j)}{\sinh \alpha \cdot \sinh \alpha(N+1)} & \text{if } k \leq j \\ \frac{\sinh \alpha j \cdot \sinh \alpha(N+1-k)}{\sinh \alpha \cdot \sinh \alpha(N+1)} & \text{if } k \geq j \end{cases}$$

for α such that $1 + (\lambda h^2 - 2)e^\alpha + e^{2\alpha} = 0$ and $\lambda \neq \frac{4}{h^2} \sin^2 \frac{k\pi h}{2}$ with $k = 0, 1, \dots, N$.

2.2. A priori estimate

Considering problem (4), one has the following result

Proposition 1. Under the hypotheses (H1), (H2), (H3) and (H4) there exists $C > 0$ such that

$$\max_{0 \leq j \leq N+1} \|w_j\|_E \leq C \max_{0 \leq j \leq N+1} \|f_j\|_E$$

Proof. Writing

$${}^tW = (w_1, w_2, \dots, w_N)$$

then

$$W = \frac{-1}{2i\pi} \int_{\gamma} (A - B^2 - \lambda I)^{-1} Y d\lambda$$

where

$${}^tY = (y_1, y_2, \dots, y_N)$$

and

$$y_j = \sum_{k=0}^{N+1} K_{jk}^\alpha e^{-B(j-k)h} f_k.$$

From (H2) one has

$$\exists K > 0 / \forall \lambda \geq 0 \quad \|(A - B^2 - \lambda J)^{-1}\|_{L(E)} \leq \frac{K}{1 + \lambda}$$

and from (H1)

$$\|e^{-B(j-k)h}\| \leq K.$$

On the other hand

$$\begin{aligned} & \sum_{k=0}^{N+1} |K_{jk}^\alpha| \\ & \leq \frac{h^2 |\sinh \alpha(N+1-j)|}{|\sinh \alpha \sinh \alpha(N+1)|} \sum_{k=0}^j |\sinh \alpha k| \\ & + \frac{h^2 |\sinh \alpha(j)|}{|\sinh \alpha \sinh \alpha(N+1)|} \sum_{k=j+1}^{N+1} |\sinh \alpha(N+1-k)| \\ & \leq \frac{h^2 \sinh \operatorname{Re} \alpha(N+1-j) \sinh \operatorname{Re}(\alpha/2)(j+1) \cosh \operatorname{Re}(\alpha/2)j}{|\sinh \alpha \sinh \alpha(N+1)| \sinh \operatorname{Re}(\alpha/2)} \\ & + \frac{h^2 \sinh \operatorname{Re} \alpha(j) \sinh \operatorname{Re}(\alpha/2)(N-j+1) \cosh \operatorname{Re}(\alpha/2)(N-j)}{|\sinh \alpha \sinh \alpha(N+1)| \sinh \operatorname{Re}(\alpha/2)} \\ & \leq \frac{h^2 \sinh \operatorname{Re} \alpha(N+2)}{|\sinh \alpha \sinh \alpha(N+1)| \sinh \operatorname{Re}(\alpha/2)} \\ & \leq \frac{h^2 \sinh \operatorname{Re}(\alpha/2)(N+2) \cosh \operatorname{Re}(\alpha/2)(N+2)}{|\sinh \alpha| |\sinh(\alpha/2)(N+1)| |\cosh(\alpha/2)(N+1)| \sinh \operatorname{Re}(\alpha/2)} \\ & \leq \frac{h^2}{|\sinh \alpha|} \left(\frac{\cosh \operatorname{Re}(\alpha/2)}{\sinh \operatorname{Re}(\alpha/2)} + \frac{\cosh \operatorname{Re}(\alpha/2)(N+1)}{\sinh \operatorname{Re}(\alpha/2)(N+1)} \right) \\ & \leq \frac{Kh^2 |4 - \lambda h^2|}{|\sinh \alpha| |(\lambda^2 h^4 - 4\lambda h^2)^{1/2}|} \\ & \leq \frac{K}{\lambda}. \end{aligned}$$

2.3. Stability and convergence

Considering problems (1)-(2) and (3), putting $\varepsilon_j = u(jh) - v_j$ we have the result of stability and convergence

Theorem 2. Under the hypotheses (H1), (H2), (H3) and (H4); $\varphi, \psi \in D(L)$ (with $L = A + B^2$) such that $A\varphi, A\psi, f(0), f(1)$ belong to $D_L(\theta, +\infty)$ and $B\varphi, B\psi$ belong to $D_L(\theta + 1/2, +\infty)$, $f \in C^{2\theta}([0, 1]; E)$ with $\theta \in]0, \frac{1}{2}[$ we have

$$\max_{0 \leq j \leq N+1} \|\varepsilon_j\|_E \leq Ch^{2\theta} \|f\|_{C^{2\theta}([0, 1]; E)}$$

Proof. The hypothesis allows us to use the result of Labbas-ElHaial [5] that gives us the existence, the unicity and maximal regularity of the solution of problem (1), (2). Remark that ε_j verifies the problem

$$\begin{cases} \frac{\varepsilon_{j+1} - 2\varepsilon_j + \varepsilon_{j-1}}{h^2} + 2B\left(\frac{\varepsilon_{j+1} - \varepsilon_j}{h}\right) + A\varepsilon_j = g_j - f_j & 1 \leq j \leq N \\ \varepsilon_0 = 0 & \varepsilon_{N+1} = 0 \end{cases}$$

with

$$g_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + 2B\left(\frac{u_{j+1} - u_j}{h}\right) + Au_j$$

then ε_j verifies the a priori estimate

$$\max_{0 \leq j \leq N+1} \|\varepsilon_j\|_E \leq C \max_{0 \leq j \leq N+1} \|g_j - f_j\|_E$$

Replacing respectively g_j and f_j , we have

$$g_j - f_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + 2B\left(\frac{u_{j+1} - u_j}{h}\right) + Au_j - (u''(jh) + 2Bu'(jh) + Au(jh))$$

then

$$g_j - f_j = \frac{1}{2}u''(jh + \theta_1 h) + \frac{1}{2}u''(jh + \theta_2 h) - u''(jh) + 2Bu'(jh + \theta_2 h) - 2Bu'(jh)$$

and using the holderianity of u'' and Bu' (see [5]), we obtain the result.

3. APPLICATION OF THE INFINITE MATRIX THEORY TO THE STUDY OF THE FINITE DIFFERENCES

3.1. Recall on infinite matrices

Let $M = (a_{j,q})_{j,q \geq 1}$ be an infinite matrix whose coefficients are reals or complex numbers. We set

$$S_1 = \left\{ (a_{j,q})_{j,k \geq 1} / \sup_{j \geq 1} \left(\sum_{q=1}^{\infty} |a_{j,q}| \right) < \infty \right\}. \quad (5)$$

From [4] we know that S_1 is a Banach algebra with respect to the norm

$$\|M\|_{S_1} = \sup_{j \geq 1} \left(\sum_{q=1}^{\infty} |a_{j,q}| \right). \tag{6}$$

By the same way we define the one-column infinite matrix $X = (x_j)_{j \geq 1}$, where x_j is a scalar for every j , and the well-known vector space

$$l^\infty = \left\{ (x_j)_{j \geq 1} / \sup_{j \geq 1} |x_j| < \infty \right\} \tag{7}$$

normed by $\|X\|_{l^\infty} = \sup_{j \geq 1} |x_j|$. We are lead to study infinite linear systems of the form

$$\sum_{q=1}^{+\infty} a_{j,q} x_q = y_j \quad j = 1, 2, \dots \tag{8}$$

where $(a_{j,q})_{j,q \geq 1}$ and $(y_j)_{j \geq 1}$ are given and y_j is a scalar for all j . (8) is equivalent to the matrix equation

$$MX = Y \tag{9}$$

with $M = (a_{j,q})_{j,q \geq 1}$, $X = (x_j)_{j \geq 1}$ and $Y = (y_j)_{j \geq 1}$.

S_1 being a unital Banach algebra, if

$$\|I - M\|_{S_1} < 1 \tag{10}$$

M is invertible in S_1 and for all $Y \in l^\infty$, equation $MX = Y$ admits a unique solution in l^∞ , given by

$$X = \sum_{k=0}^{\infty} (I - M)^k Y.$$

In [2], [6] and [7] are given spaces generalizing the preceding ones.

3.2. Application to numerical scheme

We deal with problem (1), (2), where A and B are two reals with $A < 0$, $\varphi = \psi = 0$. From (3) we deduce that (1) and (2) are equivalent to the system

$$\begin{cases} \beta_N v_j + v_j + \alpha_N \beta_N v_{j+1} = h^2 \beta_N f_j & j = 1, 2, \dots, N, \\ v_0 = 0, \quad v_{N+1} = 0 \end{cases} \tag{11}$$

where

$$\alpha_N = 1 + 2B \frac{1}{N+1} \quad \text{and} \quad \beta_N = \frac{1}{-2 - 2B \frac{1}{N+1} + A \left(\frac{1}{N+1} \right)^2}.$$

Using infinite matrices, this system can be written under the form $M_N V_N = h^2 \beta_N F_N$, where $M_N = (a_{j,q}(N))_{j,q \geq 1}$ is defined by

$$a_{jq}(N) = \begin{cases} 1 & \text{if } q = j \text{ and } j \leq N, \\ \alpha_N \beta_N & \text{if } q = j+1 \text{ and } j \leq N-1, \\ \beta_N & \text{if } q = j-1 \text{ and } j \leq N, \\ 0 & \text{otherwise;} \end{cases}$$

and

$${}^t F_N = (f_1, f_2, \dots, f_N, 0, \dots), \quad {}^t V_N = (v_1, v_2, \dots, v_N, 0, 0, \dots).$$

We see that for each integer $N, M_N \in S_1, V_N$ and $F_N \in l^\infty$. Since $A < 0$, we deduce that if h is sufficiently small

$$\|I - M_N\|_{S_1} = |\beta_N| + |\alpha_N \beta_N| = \frac{2 - 2Bh}{2 + 2Bh - Ah^2} < 1.$$

Then, system (11) admits in l^∞ only one solution which can be written under the form

$$V_N = h^2 \sum_{n=0}^{\infty} (I - M_N)^n \beta_N F_N. \quad (12)$$

3.3. Expression of the solution V_N

In this section we shall give the expressions of v_j in which f_v is zero as $v \geq N+1$.

Consider the infinite matrix $\hat{M}_N = (\hat{a}_{jq}(N))_{j,q \geq 1}$ defined by

$$\hat{a}_{jq}(N) = \begin{cases} 1 & \text{if } q = j, \\ \alpha_N \beta_N & \text{if } q = j+1, \\ \beta_N & \text{if } q = j-1, \\ 0 & \text{otherwise;} \end{cases}$$

As we have seen in [4], we can give the expression of $(I - \hat{M}_N)^n$ according to the cases when n is odd or even. Note that the calculations are easier than in the case when \hat{M}_N is replaced by M_N . We obtain the following

Proposition 3. When j is even, the coordinates of the solution are given by

$$v_j = h^2 \sum_{p=0}^{j-1} (-1)^p \beta_N^{p+1} \left(\sum_{s=0}^p C_p^s \alpha_N^s f_{j-p+2s} \right) + h^2 \sum_{p=j}^{\infty} (-1)^p \beta_N^{p+1} \gamma_{j,p} - h^2 \sum_{i=j+1}^{\infty} \beta_N^{2i+2} \left(C_{2i+1}^{i+\frac{j}{2}} - C_{2i+1}^{i-\frac{j}{2}} \right) \alpha_N^{i-\frac{j}{2}+1} f_i \quad (13)$$

with

$$\gamma_{j,p} = \sum_{s \in [0, \frac{j}{2} - \frac{1}{2}]} (C_p^{j+s} - C_p^s) \alpha_N^{p-j-s} f_{p-j-2s} + \sum_{s=1}^j C_p^{j-s} \alpha_N^{p-j+s} f_{p-j+2s}.$$

When j is odd $j/2$ is replaced by $(j-1)/2$ in the expression of $\gamma_{j,p}$ and the third term in the sum defining v_j vanishes.

Proof. Doing as in [4] one verifies that V_N is solution of system (12). In (13), v_j can be written by the means of three sums : $\sigma_{1,j}, \sigma_{2,j}$ and $\sigma_{3,j}$ respectively. Let us proof that in the case when $j = 2k, k \geq 1$, we have for all $j \in \{1, \dots, N\}$

$$\beta_N v_{j-1} + v_j + \alpha_N \beta_N v_{j+1} = h^2 \beta_N f_j.$$

The sum $\beta_N \sigma_{1,j-1} + \sigma_{1,j} + \alpha_N \beta_N \sigma_{1,j+1}$ is equal to $h^2 \beta_N f_j$, since $\forall p \in \{0, 1, \dots, j-2\}$ and $0 \leq s \leq p$ the coefficient of the term $h^2 \beta_N^{p+2} f_{j-p+2s-1}$ is equal to

$$C_p^p \alpha_N^s - C_{p+1}^s \alpha_N^s + \alpha_N C_p^{s-1} \alpha_N^{s-1} = \alpha_N^s (C_p^s - C_{p+1}^s + C_p^{s-1}) = 0.$$

If $p = j-1$ the coefficient of $h^2 \beta_N f_{2s}$ in the sum $\beta_N \sigma_{2,j-1} + \sigma_{2,j} + \alpha_N \beta_N \sigma_{2,j+1}$ is

$$C_p^{j-s-1} \alpha_N^{p-j+s+1} - C_{p+1}^{j-s} \alpha_N^{p+1-j+s} + \alpha_N C_p^{j-s} \alpha_N^{p-j+s} = 0, \forall s \in \{1, 2, \dots, j-1\}.$$

If $p = j$ the coefficient of $h^2 \beta_N f_1$ is equal to

$$(C_j^{j-1} - C_j^0) \alpha_N - (C_{j+1}^j - C_{j+1}^0) \alpha_N + \alpha_N C_j^j = 0.$$

We get the coefficients of $h^2 \beta_N^{p+2} f_3, h^2 \beta_N^{p+2} f_5, \dots$ doing the sum

$$C_j^{j-s-1} \alpha_N^{s+1} - C_{j+1}^{j-s} \alpha_N^{s+1} + \alpha_N C_j^{j-s} \alpha_N^s = 0, \quad 1 \leq s \leq j-1.$$

If $p \geq j+1$ and $s \in \{1, 2, \dots, j\}$ we get

$$C_p^{j-s-1} \alpha_N^{p-j+s+1} - C_{p+1}^{j-s} \alpha_N^{p+1-j+s} + \alpha_N C_p^{j-s} \alpha_N^{p-j+s} = 0,$$

as coefficient of $h^2 \beta_N^{p+2} f_{p-j+2s+1}$ in the sum $\beta_N \sigma_{2,j-1} + \sigma_{2,j} + \alpha_N \beta_N \sigma_{2,j+1}$. For $s = j$ one gets two terms $-C_{p+1}^0 + C_p^0 = 0$ as coefficient of $h^2 \beta_N^{p+2} f_{p+j+1}$. When $0 \leq s \leq [\frac{p}{2}] - k - 1$ the coefficient of $h^2 \beta_N^{p+2} f_{p-j-2s+1}$ is

$$(C_p^{j+s+1} - C_p^s) \alpha_N^{p-j+1-s} - (C_{p+1}^{j+s} - C_{p+1}^s) \alpha_N^{p+1-j-s} + \alpha_N (C_p^{j+s} - C_p^{s-1}) \alpha_N^{p-j-s} = 0.$$

Finally the sum of the coefficients of $h^2 \beta_N^{p+2} f_1$ corresponding to even powers of $\beta_N, (p = 2i)$, in the terms $\beta_N \sigma_{2,j-1}, \sigma_{3,j}$ and $\beta_N \sigma_{2,j+1}$ is

$$-(C_p^{k+i-1} - C_p^{i-k}) \alpha_N^{p-k+i+1} - (C_{p+1}^{i+k} - C_{p+1}^{i-k}) \alpha_N^{p+1-i-k} + \alpha_N (C_p^{i+k} - C_p^{i-k-1}) \alpha_N^{p-i-k} = 0.$$

Let us give now another expression of v_j .

Proposition 4. There exist two sequences $(\sigma_j(q, s))_{j \geq 1}$ and $(\tau_j(q, s))_{j \geq 1}$ such that

$$v_j = h^2 \sum_{q \geq 0} \beta_N^{2q+1} \left(\sum_{s \geq 0} \sigma_j(q, s) f_{2s+1} \right) - h^2 \sum_{q \geq 0} \beta_N^{2q+2} \left(\sum_{s \geq 0} \tau_j(q, s) f_{2s} \right). \quad (14)$$

There exists a constant $K_B > 0$, depending only on B , such that $|\sigma_j(q, s)| \leq K_B C_{2q}^q \alpha_N^q$, $|\tau_j(q, s)| \leq K_B C_{2q+1}^q \alpha_N^q$.

Proof. When $j = 2k + 1$, the calculation gives

$$\sigma_j(q, s) = \begin{cases} C_{2q}^{q-k+s} \alpha_N^{q-k+s} & \text{if } 0 \leq q \leq k \text{ and } k - q \leq s \leq k + q, \\ (C_{2q}^{q+k-s} - C_{2q}^{q-k-s-1}) \alpha_N^{q-k+s} & \text{if } k + 1 \leq q \text{ and } 0 \leq s \leq q - k - 1, \\ C_{2q}^{q+k-s} \alpha_N^{q-k+s} & \text{if } k + 1 \leq q \text{ and } q - k \leq s \leq k + q, \end{cases} \quad (15)$$

$$\tau_j(q, s) = \begin{cases} C_{2q+1}^{q-k+s} \alpha_N^{q-k+s} & \text{if } 0 \leq q \leq k \text{ and } k - q \leq s \leq k + q - 1, \\ (C_{2q+1}^{q+k-s+1} - C_{2q+1}^{q-k-s}) \alpha_N^{q-k+s} & \text{if } k + 1 \leq q \text{ and } 0 \leq s \leq q - k, \\ C_{2q+1}^{q+k-s+1} \alpha_N^{q-k+s} & \text{if } k + 1 \leq q \text{ and } q - k + 1 \leq s \leq k + q + 1, \end{cases} \quad (16)$$

Note that $k, s \leq \frac{N-1}{2}$ in (15) and $s \leq \frac{N}{2}$ in (16). Let us show that $|\sigma_j(q, s)| \leq K_B C_{2q}^q \alpha_N^q$. First consider the case when B is negative. We have $\alpha_N \leq 1$, therefore

$$\alpha_N^{-k} = \left(1 + \frac{2}{N+1} B \right)^{-k} \leq \left(1 + \frac{2}{N+1} B \right)^{-N} = e^{-N \ln(1 + \frac{2}{N+1} B)}.$$

Since $e^{-N \ln(1 + \frac{2}{N+1} B)}$ tends to e^{-B} , as $N \rightarrow \infty$, we deduce that there exists a constant $K'_B > 0$ such that $\alpha_N^{-k} \leq K'_B$. Elsewhere s being positive, we get

$$\alpha_N^{q-k+s} \leq \alpha_N^{q-k} \leq K'_B \alpha_N^q.$$

Consider now the case when $B \geq 0$, corresponding to $\alpha_N \geq 1$. As we have seen above there exists a constant $K''_B > 0$ such that $\alpha_N^N \leq K''_B$. In fact, here α_N^N tends to e^{2B} , as $N \rightarrow \infty$. Then, we only have to prove that $\alpha_N^{s-k} \leq K''_B$, using the relations given in (15). One gets successively, if $0 \leq s \leq q + k$ then $\alpha_N^s \leq \alpha_N^{q+k}$, and

$$\alpha_N^{s-k} \leq \alpha_N^q \leq \alpha_N^N \leq K''_B.$$

When $0 \leq s \leq q - k - 1 \leq (N-1)/2$, we have $q \leq N$, hence

$$\alpha_N^s \leq \alpha_N^q \alpha_N^{-k-1} \leq \alpha_N^q \leq \alpha_N^N \leq K''_B$$

and $\alpha_N^{s-k} \leq \alpha_N^s \leq K''_B$. Finally, when $q - k \leq s \leq q + k$ and $q \geq k + 1$ we deduce that $\alpha_N^s \leq \alpha_N^{k+q}$, and

$$\alpha_N^{s-k} \leq \alpha_N^q \leq \alpha_N^N \leq K_B^n.$$

If we set $K_B = \sup(K'_B, K''_B)$, we conclude that $|\sigma_j(q, s)| \leq K_B C_{2q}^q \alpha_N^q$. We proceed similarly for $\tau_j(q, s)$.

One deduces the following property

Proposition 5. There exists a constant $K > 0$ such that

$$\|V_N\|_{s_1} = \max_{1 \leq j \leq N} |v_j| \leq \frac{K}{N+1} \sum_{v=1}^N |f_v|. \tag{17}$$

Proof. We are going to determine an upper bound of $|v_j|$ where v_j is given by (14).

We get

$$|v_j| \leq h^2 K_B \left(\sum_{q=0}^{\infty} C_{2q}^q (\sqrt{\alpha_N} \beta_N)^{2q} |\beta_N| + \sum_{q=0}^{\infty} C_{2q+1}^q |\sqrt{\alpha_N} \beta_N|^{2q+1} |\beta_N| \right) \sum_{v=1}^N |f_v| \tag{18}$$

Using the Taylor series

$$\sum_{q=0}^{\infty} C_{2q}^q (\sqrt{\alpha_N} \beta_N)^{2q} = \frac{1}{\sqrt{1 - 4\alpha_N \beta_N^2}},$$

where $|\sqrt{\alpha_N} \beta_N| < 1/2$, h being sufficiently small. We obtain

$$h^2 \left| \sum_{q \geq 0} \beta_N^{2q+1} \left(\sum_{s \geq 0} \sigma_j(q, s) f_{2s+1} \right) \right| \leq h^2 |\beta_N| K_B \frac{|-2 - 2Bh + Ah^2|}{2h \sqrt{B^2 - A - ABh + \frac{A^2}{4} h^2}} \sum_{v=1}^N |f_v|. \tag{19}$$

Since $B^2 - A > 0$, this inequality is true as $h \rightarrow 0$. Doing analogous calculations for the second sum defining v_j in (14), we get

$$|\tau_j(q, s)| \leq 2K_B C_{2q}^q \alpha_N^q.$$

We deduce that there exists $K > 0$ for which (17) is satisfied.

3.4. Case when A is an operator and $B = 0$

When $B = 0$, the result has been generalized to the case when A is a closed linear operator, whose domain D_A is included in a Banach space G . A satisfies the unique hypothesis of ellipticity (H2), in which $B = 0$. We obtain the second order differential equation

$$u''(t) + Au(t) = g(t) \in G, \quad \text{with } u(0) = u(1) = 0.$$

As we have seen in subsection 3.3, the explicit calculation of the solution V_N permits us to show the following result given in [6].

Proposition 6. There exists $K > 0$ such that

$$\|V_N\|_{s_1} \leq \frac{K}{N+1} \sum_{v=1}^N \|f_v\|_G.$$

Indeed, here $\alpha_N = I$ and $\beta_N = (-2I + Ah^2)^{-1}$. We have

$$\|v_j\|_G \leq h^2 \left(\sum_{q=0}^{\infty} C_{2q}^q \|\beta_N\|_{L(G)}^{2q+1} + \sum_{q=0}^{\infty} C_{2q+1}^q \|\beta_N\|_{L(G)}^{2q+2} \right) \sum_{v=1}^N \|f_v\|_G,$$

where

$$\|\beta_N\|_{L(G)} \leq (N+1)^2 \|A - 2(N+1)^2 I\|_{L(G)}^{-1} \leq \frac{(N+1)^2}{1+2(N+1)^2} < \frac{1}{2}.$$

Using the Taylor series

$$\sum_{q=0}^{\infty} C_{2q}^q \|\beta_N\|_{L(G)}^{2q} = \frac{1}{\sqrt{1-4\|\beta_N\|_{L(G)}^2}},$$

we deduce that

$$h^2 \left| \sum_{q \geq 0} \beta_N^{2q+1} \left(\sum_{s \geq 0} \sigma_j(q, s) f_{2s+1} \right) \right| \leq \frac{3}{4} \frac{1}{(N+1)} \sum_{s=1}^N \|f_s\|_G,$$

which gives the conclusion.

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