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ABELIAN COVERINGS OF REGULAR HYPERMAPS OF TYPES {4,4,4} AND {3,3,4} OF GENUS 2*

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ABSTRACT

In this paper, we obtain elementary abelian coverings of the regular hypermaps of genus 2 corresponding to the orientation-preserving automorphism groups $SL_2(3)$, the special linear group of order 24, and Q_8, the quaternion group of order 8. We also determine the reflexibility of these coverings.

KEYWORDS

Abelian Covering, Map, Hypermap, Reflexibility, Chiral Pair

1. INTRODUCTION

In [1] Azevedo and Jones showed that there are 43 regular hypermaps of genus 2, of which 10 are maps. Another 20 regular hypermaps can be obtained from these regular maps. The remaining 13 regular hypermaps are not associates of these maps: they are 5 regular hypermaps and their associates. In [11] Kazaz obtained the representations ρ_1 and characters τ_1 on the first integer homology group $H_1(S, \mathbb{Z}) \cong N/N' \cong \mathbb{Z}^4$ of the orientation-preserving automorphism groups G of regular hypermaps of genus 2, where S is the Riemann surface underlying the regular hypermaps of genus 2, N is a normal subgroup of the triangle group $\Delta = \Delta(l, m, n)$ and N' is the commutator subgroup of N.

On the other hand, the universal coefficient theorem [5] implies that $H_1(S;R) = H_1 \otimes_{\mathbb{Z}} R$ for any commutative ring R of coefficients, so one can view τ_1 as giving the character of the representation ρ_1 of G on $H_1(S;R)$. If we take $R = \mathbb{Z}_q$ (q prime) as the ring of coefficients, we obtain the reduction of ρ_1 mod

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q, that is, the representation of G on the homology group $H_1(S;\ \mathbb{Z}_q) \cong N/N'N^q \cong \mathbb{Z}_q^4$ of the underlying surface S. Then we use this representation to construct finite abelian coverings of a given genus 2 regular hypermap H of type $\{l,m,n\}$ corresponding to hypermap-subgroup $N \trianglelefteq \Delta = \Delta(l,m,n)$: any G-submodule of $\mathbb{Z}_q^4$ corresponds to a normal subgroup M of the triangle group $\Delta = \Delta(l,m,n)$ lying between N and $N'N^q$, so it gives a regular unbranched covering $X/M \to X/N = S$, with the elementary abelian q-group N/M as the group of covering transformations. Since $M \trianglelefteq \Delta$, this also corresponds to a regular hypermaps $\hat{H}$ which covers H of type $\{l,m,n\}$ and genus $\hat{g}$ with Aut $\hat{H}$ $\cong \Delta/M$. If a G-invariant submodule of $H_1(S;\ \mathbb{Z}_q)$ has codimension d then it gives a q^d-sheeted regular covering of the hypermap, since q^d is the index of normal subgroup corresponding to this submodule.

We also use these representations and characters to study the reflexibility of these coverings. It is well-known that a regular hypermap H of type $\{l,m,n\}$ corresponding to a normal subgroup $N \trianglelefteq \Delta(l,m,n)$ is reflexible if and only if N is also normal in the extended triangle group,

$$\Delta^*(l,m,n) = \langle r_1, r_2, r_3 \mid r_1^2 = r_2^2 = r_3^2 = 1, (r_2r_3)^l = (r_3r_1)^m = (r_1r_2)^n = 1 \rangle$$

which contains $\Delta = \langle r_1r_2, r_2r_3 \rangle$ with index 2; in this case $\Delta^*/N \cong \mathrm{Aut}^*\ H$, including orientation-reversing automorphism. It follows that H is reflexible if and only if $N^{r_i} = N$ for some i. Hence H is reflexible if and only if Aut H has an automorphism (induced by r_3) which inverts g_1 (the image of r_2r_3) and g_2 (the image of r_3r_1). Jones and Azevedo, in [1], showed that the regular hypermaps of genus 2 are all reflexible, that is, each has an additional orientation-reversing automorphism.

Now suppose that $N' \le M \le N \le \Delta \trianglelefteq \Delta^*$, where $M \trianglelefteq \Delta$ and $N \trianglelefteq \Delta^*$, so the hypermap, say H_M corresponding to M is regular and H_N corresponding to N is reflexible. Then as Δ/N has an automorphism which inverts g_1 and g_2, it follows that H_M is reflexible if and only if this lifts to an automorphism of Δ/M. If it does, it induces an automorphism of N/M. Since $N' \le M$ (so that N/M is abelian), the character of Δ/N on N/M must be invariant under this automorphism, say α, so whenever an irreducible character χ is a summand, then so is the conjugate character χ^α given by $\chi^\alpha(g) = \chi(\alpha(g))$ for every $g \in G$ with the same multiplicity, or equivalently the corresponding G-submodule W of $\mathbb{Z}_q^4$ is invariant under

α_1 (that is, $\alpha_1 W = W$, where α_1 is the matrix of the orientation-reversing automorphism α on the homology group [11]). If it is not invariant then the corresponding hypermap is called *chiral*; in this case the corresponding normal subgroup is not normal in the extended triangle group, thus M has another conjugate $M^g \neq M(g \in \Delta^* \setminus \Delta)$, so M and M^g correspond to two chiral hypermaps of type $\{l, m, n\}$, that is, there is a chiral pair.

If $M = N' N^q$ then M is a characteristic subgroup of N, so that $M \trianglelefteq \Delta^*$, thus H_M is reflexible. Thus, in the case of genus 2, we always have two reflexible hypermaps corresponding to N and $N' N^q$ (also corresponding to $H_1 \cong \mathbb{Z}_q^4$ and the 0-submodule). Any reflexible hypermap which does not correspond to N or $N' N^q$ is called a *proper* reflexible hypermap, and unless explicitly stated otherwise, we will always consider the proper reflexible hypermaps.

2. REGULAR MAPS AND HYPERMAPS

In this chapter we define maps and hypermaps, more importantly regular maps and regular hypermaps, and indicate the related results [2], [3], [4], [7], [8], [9], [10].

A *(topological) map* M is an imbedding (without crossings) of a finite connected graph G onto a surface S which is compact, connected, orientable and without boundary, where each face of M (the connected components of $S \setminus G$) is homeomorphic to an open disc. Then S is homeomorphic to a surface S_g consisting a sphere with g handles attached, for some integer $g \geq 0$; and we call g the *genus* of M [8].

We may describe a map by means of permutations. We define a *dart* of M to be a pair consisting of an edge and an incident vertex, and draw it as an arrow on the edge towards the vertex. We let Ω be the set of all darts; i.e. $\Omega = \{(e, v) \mid e \in E,$ $v \in V,$ $e \cap v = v$, where E and V are the sets of edges and vertices of M, respectively. We define a permutation x on Ω by interchanging two darts on an edge, so that x is a product of transpositions. The orientation of S induces cyclic permutations of the darts pointing to each vertex v, and these form the disjoint cycles of a permutation y of Ω. Similarly, by following the orientation of S around each face of M, we obtain a permutation z of Ω. Then these give the relations, $x^2 = y^m = z^n = xyz = 1$, where m and n are the least common multiple of the valencies of the vertices and faces. Now let $G = \langle x, y \rangle$ be a subgroup of S^{Ω}, the group of all permutations of Ω. Since G is connected, it follows that G is transitive [9].

An *algebraic map* A is a quadruple (G,Ω,x,y), where Ω is a set and x,y are permutations of Ω such that $x^2=1$ and $G=<x, y>$ is transitive on Ω. We say that A has *type* $\{m,n$ if y and xy^{-1} have orders m and n, respectively.

Let $\Delta=\Delta(2,m,n)=<X,Y,Z|X^2=Y^m=Z^n=XYZ=1>$ be the triangle group and let M be a map of type $\{m,n$. Then there is an epimorphism $\theta:\Delta\rightarrow G$ given by $X\rightarrow x, Y\rightarrow y, Z\rightarrow z$, so Δ has a transitive action on Ω. If M is the stabilizer of a dart α in this action of Δ, that is, $M = \theta^{-1}(G_\alpha)$, where $G_\alpha = \{g\in G \mid \alpha g=\alpha\}$, then we call M a *map–subgroup* for M

The study of maps is closely related to the study of subgroups of certain triangle groups $\Delta(2,m,n)$. Jones and Singerman, in [9], showed that there is a natural correspondence between maps and Schreier coset graphs for the subgroups of the triangle groups $\Delta(2,m,n)$, furthermore every subgroup of a triangle group is a map-subgroup for some map.

If maps M_i $(i=1,2)$ are given by imbeddings of graphs G_i, with vertex sets V_i, in surfaces S_i, then a *morphism* ϕ: $M_1 \rightarrow M_2$ is an orientation-preserving surface covering (possibly branched) ϕ: $S_1\rightarrow S_2$ such that

(i) ϕ^{-1} $(G_2)=G_1$ and ϕ^{-1} $(V_2)=V_1$,

(ii) all branched points have finite order.

We say that a topological map M_1 *covers* M_2 if there is a morphism ϕ: $M_1\rightarrow M_2$.

We can also define a morphism between algebraic maps: given algebraic maps A_i $=(G_i, \Omega_i, x, y)$ $(i = 1,2)$, a *morphism* between A_1 and A_2 is a pair (σ,τ) of functions $\sigma: G_1 \rightarrow G_2$, τ : $\Omega_1 \rightarrow \Omega_2$, where σ is a group homomorphism, $x_1\tau = x_2$, $y_1\tau = y_2$, and the following diagram commutes (that is, $(\alpha g)\tau = (\alpha\tau)(g\phi)$ for all $g\in G$ and $\alpha\in\Omega_1$) (the horizontal arrows represent group actions).

$$
\begin{array}{ccc}
\Omega_1 \times G_1 & \longrightarrow & \Omega_1 \\
(\tau,\sigma)\ \Big\downarrow & & \Big\downarrow\ \tau \\
\Omega_2 \times G_2 & \longrightarrow & \Omega_2
\end{array}
$$

We say that A_1 *covers* A_2 if there is a morphism $A_1\rightarrow A_2$.
A morphism ϕ: $M_1 \rightarrow M_2$ is an *isomorphism* if it is one-to-one.

Theorem 1. *If* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are algebraic maps of type* $\{m,n\}$, *then* $\mathcal{A}_1$ *covers* $\mathcal{A}_2$ *if and only if we can find map-subgroups* $M_i \leq \Delta(2,m,n)$ *for* $\mathcal{A}_i$ $(i=1,2)$, *respectively, with* $M_1 \leq M_2$.

A map $\mathcal{M}$ is called *regular* if Aut $\mathcal{M}$ is transitive on Ω.

Theorem 2. *Let* $\mathcal{M}$ *be a map with map-subgroup* $M \leq \Delta(2,m,n)$ *and associated algebraic map* (G,Ω,x,y). *Then the following conditions are equivalent:*

(i) $\mathcal{M}$ is regular;

(ii) G is a regular permutation group on Ω, that is, $G_\alpha = 1$, for some $\alpha \in \Omega$;

(iii) M is a normal subgroup of Δ.

When these conditions hold, then we have Aut $\mathcal{M} \cong G \cong \Delta / M$.

Suppose that $\mathcal{M}_1$ is a covering map of $\mathcal{M}_2$. Then by Theorem 1 there is a triangle group Δ and map-subgroups M_1, M_2 for $\mathcal{M}_1$, $\mathcal{M}_2$ respectively, with $M_1 \leq M_2 \leq \Delta$. We call the cover *finite* if M_1 has finite index in M_2. We call the cover *regular* if $M_i \trianglelefteq \Delta$ with $M_1 \trianglelefteq M_2$.

We say $\mathcal{M}$ is *finite* if Ω is finite (or equivalently if S is compact, connected, orientable surface and without boundary) and $\mathcal{M}$ is said to be of *finite type* if m and n are finite.

Essentially an algebraic map is a quadruple (G,Ω,x,y), where G is a group generated by x, y with $x^2=1$, acting transitively on a set Ω. When we drop the condition that $x^2=1$, then the result in an algebraic (or topological) hypermap, that is, an edge is allowed to intersect any number of vertices. Thus a map is simply a hypermap.

A *topological hypermap* $\mathcal{H}$ on a compact orientable surface S and without boundary is a triple (S,R,A) where R and A are closed subsets of S such that:

(i) $B = R \cap A$ is a non-empty finite set,

(ii) $R \cup A$ is connected,

(iii) each of component of R is homeomorphic to a closed disc and each component of A is homeomorphic to a closed disc,

(iv) each component of $S \setminus (R \cup A)$ is homeomorphic to an open disc.

The components of R, A and $S \setminus (R \cup A)$ are called *hyperedges, hypervertices*, and *hyperfaces*, respectively. Finally, the elements of B are called *bits*.

If l and m are the least common multiples of the number of bits in the hyperedges and in the hypervertices, respectively, and n is the least common multiple of the number of hyperedges incident with each hyperface then the hypermap is said to be of *type* $\{l,m,n\}$. The *genus* of $\mathcal{H}$ is the genus of S.

We also define a topological hypermap by means af permutations: a hypermap H consists of a set of objects B, which we have already called the bits of H, together with two permutations x and y on B such that $G=<x,y>$ is a transitive permutation group on B. The cycles of x and y correspond to hyperedges, respectively, while the cycles of $z=(xy^{-1})$ correspond to hyperfaces.

An *algebric hypermap* is a quadruple $A=(G,B,x,y)$ where B is a set and x,y are two permutations of B such that $G=<x,y>$ is transitive on B. If x,y and xy have orders l,m and n, respectively, then we say that A has *type* $\{l,m,n$.

If H has type $\{l,m,n\}$ then there is an natural epimorphism $\theta: \Delta(l,m,n) \rightarrow G$ given by $X \rightarrow x$, $Y \rightarrow y$ and $Z \rightarrow (xy)^{-1}$. If $G_\alpha = \{g \in G \mid \alpha g = \alpha$ for any $\alpha \in B$ then $H = \theta^{-1}(G_\alpha)$ is called the *hypermap-subgroup* for H.

Let H_1 and H_2 be two topological hypermaps with underlying surfaces S_1 and S_2 and underlying graphs G_1 and G_2. A *topological morphism* ϕ: $H_1 \rightarrow H_2$ is an orientation-preserving surface covering (possibly branched) ϕ: $S_1 \rightarrow S_2$ such that

(i) $\phi^{-1}(G_2)=G_1$.

(ii) $\phi^{-1}(V_{H_2})=V_{H_1}$,

(iii) $\phi^{-1}(E_{H_2})=E_{H_1}$,

where V_{H_i}, E_{H_i} are the sets of hypervertices and hyperedges of H_i (i =1, 2).

Let $A_1 = (G_1, B_1, x_1, y_1)$ and $A_2 = (G_2, B_2, x_2, y_2)$ be two algebraic hypermaps. An algebraic morphism form A_1 to A_2 is a pair (σ, f) formed by a homomorphism $\sigma: G_1 \rightarrow G_2$ which satisfies $x_1 = x_2\sigma$ and $y_1 = y_2\sigma$ and a function $f: B_1 \rightarrow B_2$ which makes the following diagram commute, that is, $(\alpha g)f = (\alpha f)(g\sigma)$ for all $g \in G$ and $\alpha \in B_1$:

$$\begin{array}{ccc} B_1 \times G_1 & \longrightarrow & B_1 \\ {\scriptstyle (\sigma, f)} \downarrow & & \downarrow {\scriptstyle f} \\ B_2 \times G_2 & \longrightarrow & B_2 \end{array}$$

Since G_1 is transitive, σ is an epimorphism and f is onto. We also note that (σ, f) is an isomorphism if σ is a group isomorphism and f is a bijection.

If $B_1=B_2=B$ and $G_1=G_2=G=<x,y>$ then (σ,f) is called an *automorphism* of $\mathcal{A}=(G,B,x,y)$. In this case σ fixes the generators x,y so that σ is the identity, and thus an automorphism is determined by the bijection $f:B\to B$. The commutativity of the diagram implies that f commutes with all $g\in G$, so we see that the automorphisms of an algebraic hypermap $\mathcal{A}$ form a group Aut $\mathcal{A}$.

Theorem 3. *If $\mathcal{A}$ is a algebraic hypermap with hypermap-subgroup $H\leq\Delta$ then* Aut $\mathcal{A}\cong N_\Delta(H)/H$.

We say that $\mathcal{H}$ is regular if Aut $\mathcal{H}$ is transitive on B, the set of bits of $\mathcal{H}$.

Theorem 4. *Let $\mathcal{H}=(G,B,x,y)$ be a hypermap with hypermap-subgroup $H\leq\Delta(l,m,n)$. Then the following statements are equivalent:*

(i) $\mathcal{H}=(G,B,x,y)$ *is regular;*

(ii) G *acts semi-regularly on* B, *i.e.* $G_\alpha=1$, *for any* $\alpha\in B$;

(iii) $H\trianglelefteq\Delta$;

(iv) Aut $\mathcal{H}\cong G\cong\Delta/H$;

(v) $|G|=|B|$.

We say that $\mathcal{H}_1$ *covers* $\mathcal{H}_2$ if there is a morphism $\mathcal{H}_1\to\mathcal{H}_2$.

Theorem 5. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypermaps of type $\{l,m,n\}$. Then $\mathcal{H}_1$ covers $\mathcal{H}_2$ if and only if there exist hypermap-subgroups $H_i\leq\Delta(l,m,n)$ for $\mathcal{H}_i$ $(i=1,2)$ with $H_1\leq H_2$.*

Theorem 6. *Every subgroup $H\leq\Delta$ is a hypermap-subgroup for some topological hypermap.*

When a hypermap $\mathcal{H}$ is regular then Aut $\mathcal{H}\cong G\cong\Delta/H$. If $\mathcal{H}_1$ is a covering hypermap of $\mathcal{H}_2$, then by Theorem 5 there is a triangle group Δ and hypermap-subgroups H_1 and H_2 for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, with $H_1\leq H_2\leq\Delta$. Now if $H_i\trianglelefteq\Delta$ with $H_1\trianglelefteq H_2$ then the cover is said to be *regular*.

The regular hypermaps on the sphere and torus have been determined by Corn and Singerman in [3]; in each case there are infinitely many regular hypermaps. On the other hand, it is known that the number of regular hypermaps of each genus $g\geq 2$ is finite [8]. Moreover, all regular hypermaps $\mathcal{H}$ of genus 2 are known. If $\mathcal{H}$ is a map then the possibilities for its type $\{m,n\}$ and automorphism group Aut $\mathcal{H}$ are given by Coxeter and Moser in [4, p.140]. If $\mathcal{H}$ is not a map then the possibilities for its type $\{l,m,n\}$ and automorphism group Aut $\mathcal{H}$ are given by Corn and Singerman [3]. In fact, Azevedo and Jones, in [1], have completed full results by enumerating, describing and constructing all these hypermaps and specifying their full automorphism groups Aut* $\mathcal{H}$ including orientation-reversing automorphisms.

3. ELEMENTARY ABELIAN COVERINGS

Now we will find elementary abelian coverings of regular hypermaps of genus 2 corresponding to inclusion of triangle groups: $\Delta(4,4,4) \trianglelefteq \Delta(3,3,4)$ [14], and hence the automorphism groups $Q_8 \trianglelefteq SL_2(3)$.

These groups have the following relationships:

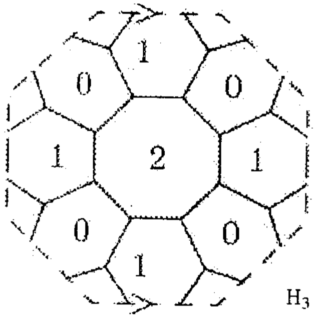
(I). $Q_8 = \langle k, l \mid k^4 = 1,\ k^2 = l^2,\ klk = l \rangle$, where $k = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$, $l = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$

and $l = d^{-1}ed$, $d = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$, $e = k = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$.

(II). $SL_2(3) = \langle d, e \mid d^3 = e^4 = 1,\ ed^2e = ded \rangle$.

Figure 1 shows the hypermaps corresponding to this inclusion: the regions labelled 0, 1, and 2 represent hypervertices, hyperedges, and hyperfaces, respectively.

Aut H₃ = Q_8



Aut H₅ = $SL_2(3)$

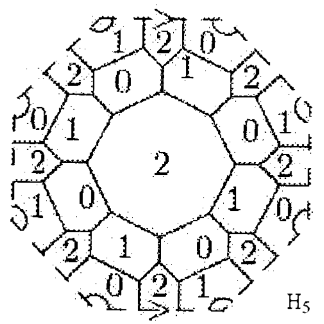


Figure 1: The regular hypermaps H_3, H_5

(I). Let $\Delta = \Delta(4,4,4) = \langle x, y, z \mid x^4 = y^4 = z^4 = xyz = 1 \rangle$ and $G = Q_8 = \langle k, l \mid k^4 = 1,\ k^2 = l^2,\ klk = 1 \rangle$. It was shown in [11] that the homology representation and character of G are

$$\rho_1 : k \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = K,\quad l \mapsto \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix} = L,$$

and

$$\tau_1 = 2\chi_5,$$

where χ_5 is the 2-dimensional irreducible complex character of degree 2 of G given by $\chi_5(1) = 2$, $\chi_5(k^2) = -2$, $\chi_5(k) = \chi_5(l) = \chi_5(lk) = 0$ (where $1, k^2, k, l, kl$ are the representatives of conjugacy classes of G).

Since $\tau_1 = 2\chi_5$, it follows that $H_1(S;\mathbb{C}) = W_1 \oplus W_2$ with $W_1 \cong_G W_2$, where W_1, W_2 are 2-dimensional G-subspaces [6]. Then we claim that there are infinitely many G-isomorphic but different submodules. Indeed, let $\theta : W_1 \mapsto W_2$ be a G-isomorphism. Let us define the function $\phi_\lambda : W_1 \to W_1 \oplus W_2$ $(\lambda \in \mathbb{C})$ by

$$\phi_\lambda : u \mapsto u + \lambda\theta(u) \quad (u \in W_1).$$

Then, it is not difficult to see that ϕ_λ is a G-homomorphism. On the other hand, $u \in Ker\,\phi_\lambda$ if and only if $u + \lambda\theta(u) = 0$ if and only if $u = 0$, since the sum $W_1 + W_2$ is direct. Thus $W_1 \cong \mathrm{Im}\phi_\lambda$. Moreover, if $\lambda \neq \mu$ then $\mathrm{Im}\phi_\lambda \neq \mathrm{Im}\phi_\mu$. Thus we have infinitely many G-submodules $\mathrm{Im}\phi_\lambda$ of the required form.

Furthermore, if V is a non-zero G-module over a finite field $GF(p)$ (p is a prime) and also is a direct sum of two G-isomorphic submodules, that is, $V = U_1 \oplus U_2$ with $U_1 \cong_G U_2$, then there exist $p+1$ G-isomorphic, but different G-submodules, namely

$$U(\lambda) = \{u + \lambda v \mid u \in U_1, v = \theta(u) \in U_2\},$$

where $\lambda \in GF(p)$ with $U(0) = U_1$, and $U(\infty) = U_2$.

We will now find these G-invariant submodules. Eigenvalues of

$$K = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

are the roots of the characteristic polynomial $(x^2+1)^2$ for K, so they are $\lambda_1 = i$ and $\lambda_2 = -i$, where $i^2 = 1$. Corresponding eigenvectors are

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ -i \\ 0 \end{pmatrix},\ w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -i \end{pmatrix}, \quad \text{and } w_3 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix},\ w_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix},$$

respectively.

Since L has the same characteristic polynomial as K, and the same eigenvalues, it follows that corresponding eigenvectors are

$$v_1 = \begin{pmatrix} i-1 \\ 1 \\ 0 \\ i \end{pmatrix},\ v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ i-1 \end{pmatrix}, \quad \text{and } v_3 = \begin{pmatrix} -(i+1) \\ 1 \\ 0 \\ -i \end{pmatrix},\ v_4 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ -(i+1) \end{pmatrix},$$

respectively.

Then one can easily check that $U_1 = < u_1 := w_1, u_2 := v_1 >$ and $U_2 = < w_3, v_3 >$ are 2-dimensional k- and l-invariant, so G-invariant submodules with $H_1(S;\mathbb{C}) \cong \mathbb{C}^4 \cong U_1 \oplus U_2$. The matrices for k and l are

$$K_1 = \begin{pmatrix} i & 0 \\ -(i+1) & -i \end{pmatrix},\ L_1 = \begin{pmatrix} -i & 1-i \\ 0 & i \end{pmatrix} \text{ on } U_1;$$

and

$$K_2 = \begin{pmatrix} -i & 0 \\ i-1 & i \end{pmatrix},\ L_2 = \begin{pmatrix} i & 1-i \\ 0 & -i \end{pmatrix} \text{ on } U_2,$$

so

$$\rho_1 : k \mapsto \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix},\ l \mapsto \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}.$$

Now two G-submodules V and W are isomorphic if and only if there is a basis B_1 of V and a basis B_2 of W such that

$$[g]_{B_1} = [g]_{B_2}$$

for all $g \in G$, [6], where $[g]_B$ denotes the matrix of the endomorphism $v \mapsto gv$ of V, relative to the basis B. Thus we can find a new basis, say $\{u_3 = xw_3 + yv_3,\ u_4 = aw_3 + bv_3\}$ $(x, y, a, b \in \mathbb{C})$ for U_2 so that $K_1 = K_2$ and $L_1 = L_2$, that is, Ku_3, Ku_4, Lu_3 and Lu_4 must be equal to

$$iu_3,\ -(i+1)u_3 - iu_4,\ -iu_3 + (1-i)u_4 \text{ and } iu_4,$$

respectively. Then as $Ku_3 = iu_3$, hat is,

$$Ku_3=\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}x-y(i+1)\\y\\ix\\-iy\end{pmatrix}=\begin{pmatrix}-ix\\iy\\x-y(i+1)\\y\end{pmatrix}$$

$$=iu_3$$

$$=i\begin{pmatrix}x-y(i+1)\\y\\ix\\-iy\end{pmatrix},$$

it follows that $y=x(1-i)$. Similarly,

$$Ku_4=\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}a-b(i+1)\\b\\ia\\-ib\end{pmatrix}=\begin{pmatrix}-ia\\ib\\a-b(i+1)\\b\end{pmatrix}$$

$$=-(i+1)u_3-iu_4$$

$$=-(i+1)\begin{pmatrix}x-y(i+1)\\y\\ix\\-iy\end{pmatrix}-i\begin{pmatrix}a-b(i+1)\\b\\ia\\-ib\end{pmatrix}.$$

This yields

$$\begin{aligned}-x(i+1)+2iy-ai+b(i-1)&=-ai,\\ -y(i+1)-bi&=bi,\\ -x(i-1)+a&=a-b(i+1),\\ y(i-1)-b&=b.\end{aligned}$$

Then we obtain $x=-bi$ and $y=x(1-i)$.

In the same way one can find from $Lu_3=-iu_3+(1-i)u_4$ (or $Lu_4=iu_4$) that $a=b(i+1)$. Thus we have

$$\begin{aligned}x&=-bi,\\ y&=x(1-i),\\ a&=b(i+1).\end{aligned}$$

Choosing $b=i$ gives $x=1$, $y=1-i$ and $a=i-1$. Thus we obtain

$$u_3 = \begin{pmatrix} -1 \\ 1-i \\ i \\ -(i+1) \end{pmatrix} \quad \text{and} \quad u_4 = \begin{pmatrix} 0 \\ i \\ -(i+1) \\ 1 \end{pmatrix},$$

and hence $K_1 = K_2 = \begin{pmatrix} i & 0 \\ -(i+1) & -i \end{pmatrix}$ and $L_1 = L_2 = \begin{pmatrix} -i & 1-i \\ 0 & i \end{pmatrix}$ as required.

Now we will reduce ρ_1 mod q (q prime).

If $q \equiv 1 \pmod 4$ then -1 is a quadratic residue mod q [13], so $U_1 = \langle u_1, u_2 \rangle$ and $U_2 = \langle u_3, u_4 \rangle$ are 2-dimensional G-invariant submodules of $\mathbb{Z}_q^4$ with $H_1(S; \mathbb{Z}_q) \cong \mathbb{Z}_q^4 \cong U_1 \oplus U_2$, and $U_1 \cong_G U_2$. Now we define

$$U(\lambda) = \langle u_1 + \lambda u_3, u_2 + \lambda u_4 \rangle$$

$$= \langle s_1 := \begin{pmatrix} 1-\lambda \\ \lambda(1-i) \\ -i(1-\lambda) \\ -\lambda(i+1) \end{pmatrix}, s_2 := \begin{pmatrix} i-1 \\ 1+\lambda i \\ -\lambda(i+1) \\ i+\lambda \end{pmatrix} \rangle,$$

where $\lambda \in GF(q)$, and $U(\infty) = U_2$. Thus we have $U(\lambda_i) \cong_G U(\lambda_j)$ with $\mathbb{Z}_q^4 \cong U(\lambda_i) \oplus U(\lambda_j)$ $(i \neq j)$ (it is easy to check that $U(\lambda)$ is a G-isomorphic submodule for all $\lambda \in GF(q) \cup \{\infty\}$, indeed $K s_1 = i s_1$, $K s_2 = -(i+1)s_1 - i s_2$ and $L s_1 = -i s_1 + (1-i)s_2$, $L s_2 = i s_2$, so $U(\lambda)$ has the same matrix for all λ). Thus there are $q+1$ 2-dimensional G-invariant submodules, giving $q+1$, q^2-sheeted regular unbranched coverings of type $\{4,4,4\}$ with genus $\hat{g} = 1 + q^2$ and automorphism groups $\hat{G} = \Delta / M_0 \cong \ldots \cong \Delta / M_\infty \cong \mathbb{Z}_q^2 \rtimes Q_8$, where $\rtimes$ denotes the semidirect product.

Moreover, it was shown in [11] that the matrix α_1 of the orientation-reversing automorphism is equal to

$$\alpha_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Now suppose that $\alpha_1 U(\lambda) = \alpha_1(\langle s_1, s_2 \rangle) = U(\mu) = \langle s_1', s_2' \rangle$ for some $\lambda, \mu \in GF(q) \cup \{\infty\}$. Then there exist some $x, y \in GF(q)$ such that

$$\alpha_1 s_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1-\lambda \\ \lambda(1-i) \\ i(\lambda-1) \\ -\lambda(i+1) \end{pmatrix} = \begin{pmatrix} \lambda-1 \\ -\lambda(i+1) \\ i(\lambda-1) \\ \lambda(1-i) \end{pmatrix}$$

$$= x s_1' + y s_2' = x \begin{pmatrix} 1-\mu \\ \mu(1-i) \\ i(\mu-1) \\ -\mu(i+1) \end{pmatrix} + y \begin{pmatrix} i-1 \\ 1+\mu i \\ -\mu(i+1) \\ \mu+1 \end{pmatrix}.$$

This yields

$$x(1-\mu)+y(i-1)=\lambda-1, \tag{1}$$

$$x\mu(1-i)+y(1+\mu i)=-\lambda(i+1), \tag{2}$$

$$xi(\mu-1)-y\mu(i+1)=i(\lambda-1), \tag{3}$$

$$-x\mu(i+1)+y(\mu+i)=\lambda(1-i). \tag{4}$$

Multiplying the equation (2) by i and adding it to (4) gives

$$y=-\lambda(i+1), \tag{5}$$

and multiplying the equation (1) by i and adding it to (3) gives

$$y=\frac{2i(1-\lambda)}{(i+1)(1+\mu)}, \quad \mu\neq -1, \tag{6}$$

Then equating (5) and (6) gives $\mu=-\frac{1}{\lambda}$, $\lambda\neq 0$. More calculations show that $\alpha_1 U(0)=U(\infty)$ and $\alpha_1 U(1)=U(-1)$.

Now suppose that $\lambda=\mu=-1/\lambda$, that is, $\lambda^2=-1$, so this has solutions if and only if $q\equiv 1 \pmod 4$. Thus there are two reflexible hypermaps corresponding to $U(\lambda), U(-\lambda)$ with $\lambda^2=-1$ (these are the only reflexible hypermaps) and $\frac{q-1}{2}$ chiral pairs corresponding to the pairs $\left(W(\lambda), W(-\frac{1}{\lambda})\right)$ with $\lambda\neq 0, \infty$, and $(W(0), W(\infty))$.

Let $q\equiv 3 \pmod 8$. We will show that $\mathbb{Z}_q^4$ has only $q+1$ different proper G-isomorphic 2-dimensional G-invariant submodules. Firstly, we will find two G-

isomorphic 2-dimensional G-submodules U. We claim that there exist two pairs $u_1, u_2 \in \mathbb{Z}_q^4$ of the following form

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \gamma \\ \beta \end{pmatrix}, \; u_2 = -Ku_1 = \begin{pmatrix} \gamma \\ \beta \\ -1 \\ 0 \end{pmatrix} \quad (\gamma, \beta \in GF(q))$$

spanning these G-submodules. Because Lu_1 and Lu_2 must be in U (Ku_1 and Ku_2 are already in U), it follows that there exist $x, y \in GF(q)$ such that

$$Lu_1 = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \gamma \\ \beta \end{pmatrix} = \begin{pmatrix} -(\gamma+\beta) \\ 1+\gamma \\ -1+\beta \\ 1-(\gamma+\beta) \end{pmatrix}$$

$$= xu_1 + yu_2 = \begin{pmatrix} x+\gamma y \\ \beta y \\ \gamma x - y \\ \beta x \end{pmatrix}.$$

It follows that

$$\begin{aligned} x+\gamma y &= -(\gamma+\beta), \\ \beta y &= \gamma+1, \\ \gamma x - y &= \beta - 1, \\ \beta x &= 1-(\gamma+\beta), \end{aligned}$$

and solving these equations gives

$$y = \frac{\gamma+1}{\beta}, \; x = \frac{1-(\gamma+\beta)}{\beta} \qquad (\beta \neq 0)$$

with

$$\gamma^2 + \gamma\beta + \beta^2 - \beta + 1 = 0. \tag{7}$$

If we choose $\gamma = 1$ we obtain $\beta^2 + 2 = 0$. Since -2 is a quadratic residue mod q [13], there exists such $\beta \in GF(q)$ with $\beta^2 = -2$. Hence we have two 2-dimensional G-invariant submodules, namely

$$U_1 = < u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ \beta \end{pmatrix}, \; u_2 = \begin{pmatrix} 1 \\ \beta \\ -1 \\ 0 \end{pmatrix} >$$

and

$$U_2 = < u_3' = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -\beta \end{pmatrix}, \ u_4' = \begin{pmatrix} 1 \\ -\beta \\ -1 \\ 0 \end{pmatrix} >$$

with $\mathbb{Z}_q^4 \cong U_1 \oplus U_2$. The matrix representations of k and l on these submodules are

$$K_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ L_1 = \begin{pmatrix} -1 & -\beta \\ -\beta & 1 \end{pmatrix}$$

and

$$K_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ L_2 = \begin{pmatrix} -1 & \beta \\ \beta & 1 \end{pmatrix},$$

respectively.

Now we can choose a new basis for U_2, say $\{u_3, u_4\}$, so that $K_1 = K_2$ and $L_1 = L_2$. Let

$$u_3 = x u_3' + y u_4' = \begin{pmatrix} x+y \\ -\beta y \\ x-y \\ -\beta x \end{pmatrix} \quad \text{and} \quad u_4 = a u_3' + b u_4' = \begin{pmatrix} a+b \\ -\beta b \\ a-b \\ -\beta a \end{pmatrix},$$

where $x, y, a, b \in GF(q)$. Then $K u_3$, $K u_4$, $L u_3$ and $L u_4$ must be equal to

$$-u_4, u_3, -u_3 - \beta u_4 \text{ and } -\beta u_3 + u_4,$$

respectively. Thus from $K u_3 = -u_4$, that is,

$$K u_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x+y \\ -\beta y \\ x-y \\ -\beta x \end{pmatrix} = \begin{pmatrix} -x+y \\ -\beta x \\ x+y \\ -\beta y \end{pmatrix} = -u_4 = \begin{pmatrix} -(a+b) \\ \beta b \\ -a+b \\ \beta a \end{pmatrix},$$

it follows that $x = b$, $y = -a$ with $\beta \neq 0$. Similarly,

$$L u_3 = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x+y \\ -\beta y \\ x-y \\ -\beta x \end{pmatrix} = \begin{pmatrix} -x + y\beta x + \beta y \\ 2x - \beta y \\ -x - y - \beta x + \beta y \\ 2y + \beta x \end{pmatrix}$$

$$= -u_3 - \beta u_4 = \begin{pmatrix} -x - y - \beta(a+b) \\ \beta y - 2b \\ -x - y - \beta(a-b) \\ \beta x - 2a \end{pmatrix}$$

gives $y=-a$, $x=-\beta a-b$. Thus we have $y=-a$, $x=b$ and $x=-\beta a-b$, so

$$y=-a,\ x=b,\ a=\beta b.$$

Choosing $b=1$ gives $x=1$, $y=-\beta$ and $a=\beta$. Thus we obtain

$$u_3=\begin{pmatrix}1-\beta\\-2y\\1+\beta\\-\beta\end{pmatrix},\quad u_4=\begin{pmatrix}\beta+1\\-\beta\\\beta-1\\2\end{pmatrix}$$

Now let us define

$$U(\lambda)=<u_1+\lambda u_3, u_2+\lambda u_4>$$
$$=<v_1:=\begin{pmatrix}1+\lambda(1-\beta)\\-2\lambda\\1+\lambda(1+\beta)\\\beta-\lambda\beta\end{pmatrix},\ v_2:=\begin{pmatrix}1+\lambda(1+\beta)\\\beta-\lambda\beta\\-1+\lambda(\beta-1)\\2\lambda\end{pmatrix}>,$$

where $\lambda\in GF(q)$, and $U(\infty)=U_2$. Then we have $U(\lambda_i)\cong_G U(\lambda_j)$ and $H_1\cong\mathbb{Z}_q^4\cong U(\lambda_i)\oplus U(\lambda_j)$ $(i\neq j)$. Thus there are $q+1$ isomorphic 2-dimensional G-invariant submodules, so $q+1$ q^2-sheeted regular unbranched coverings of type $\{4,4,4\}$ with genus $\hat{g}=1+q^2$ and $\hat{G}\cong\Delta/M_\lambda\cong\mathbb{Z}_q^2\rtimes Q_8$ for all $\lambda\in GF(q)\cup\{\infty\}$.

Now suppose that there exist λ, $\mu\in GF(q)$ such that

$$\alpha_1 U(\lambda)=\alpha_1(<v_1,v_2>)=U(\mu)=<v_1',v_2'>.$$

Then it follows that

$$\alpha_1 v_1=\begin{pmatrix}-1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}1+\lambda(1-\beta)\\-2\lambda\\1+\lambda(1+\beta)\\\beta-\lambda\beta\end{pmatrix}=\begin{pmatrix}-1-\lambda(1-\beta)\\\beta-\lambda\beta\\1+\lambda(1+\beta)\\-2\lambda\end{pmatrix}$$
$$=xv_1'+yv_2'=x\begin{pmatrix}1+\mu(1-\beta)\\-2\mu\\1+\mu(1+\beta)\\\beta-\mu\beta\end{pmatrix}+y\begin{pmatrix}1+\mu(1+\beta)\\\beta-\mu\beta\\-1+\mu(\beta-1)\\2\mu\end{pmatrix},$$

for some x, $y\in GF(q)$. This leads to the following equations:

$$\overbrace{(1+\mu(1-\beta))}^{A}x+\overbrace{(1+\mu(1+\beta))}^{B}y=\overbrace{-1-\lambda+\lambda\beta}^{E},\qquad(8)$$
$$\overbrace{(-2\mu)}^{C}x+\overbrace{(\beta-\mu\beta)}^{D}y=\overbrace{\beta-\lambda\beta}^{F},\qquad(9)$$

$$\overbrace{(1+\mu(1+\beta))}^{B}x-\overbrace{(1-\mu(\beta-1))}^{A}y=\overbrace{1+\lambda+\lambda\beta}^{G}, \tag{10}$$

$$\overbrace{(\beta-\mu\beta)}^{D}x-\overbrace{(-2\mu)}^{C}y=\overbrace{-2\mu}^{H}. \tag{11}$$

Multiplying the equation (8) by B and (10) by $-A$ and then adding these two equations, we obtain

$$y(A^2+B^2)=BE-AG. \tag{12}$$

Similarly, multiplying the equation (9) by D and the equation (11) by $-C$ and then adding them gives

$$y(C^2+D^2)=DF-CH. \tag{13}$$

Now we claim that $A^2+B^2\neq 0$ and $C^2+D^2\neq 0$. Assume that $A^2+B^2=0$, i.e.

$$\begin{aligned} A^2+B^2 &= (1+\mu(1-\beta))^2+(1+\mu(1+\beta))^2 \\ &= -2\mu^2+4\mu+2 \\ &= 0. \end{aligned}$$

Then the discriminant of this equation is

$$\mathrm{D} \;=32=2\cdot 16\,.$$

Since 2 is a quadratic non-residue mod q [13], D is a quadratic non-residue mod q, (we note that the product of a quadratic residue and a quadratic non-residue is a quadratic non-residue [13]), that is, the equation has no root in $GF(q)$, so $A^2+B^2\neq 0$. Similarly, if

$$C^2+D^2=(-2\mu)^2+(\beta-\mu\beta)^2=2\mu^2+4\mu-2=0,$$

then we obtain

$$\mathrm{D} = 32 = 2\cdot 16\,.$$

Thus $C^2+D^2\neq 0$. Therefore from the equations (12) and (13) we obtain

$$y=\frac{BE-AG}{A^2+B^2}=\frac{DF-CH}{C^2+D^2}, \tag{14}$$

and putting the values of A, B, C, D, E, F, G and H in (14) gives

$$\frac{-2-2\lambda-2\mu-6\lambda\mu}{-2\mu^2+4\mu+2}=\frac{-2+2\lambda+2\mu-6\lambda\mu}{2\mu^2+4\mu-2}.$$

Then it follows that

$$-6\lambda\mu^3-6\mu^2+2\lambda\mu+2=\left(\mu+\frac{1}{\lambda}\right)\left(-3\lambda\mu^2+\lambda\right)=0,$$

and so

$$\left(\mu+\frac{1}{\lambda}\right)\left(-3\mu^2+1\right)=0 \quad (\lambda\neq 0). \tag{15}$$

Now the same calculations for $v_2\in U(\lambda)$ give

$$\lambda\mu^3+\mu^2=\mu^2(\mu\lambda+1)=0. \tag{16}$$

Then it follows from (15) and (16) that $\mu=-\frac{1}{\lambda}$ $(\lambda\neq 0)$ with $x=\lambda\beta$, $y=-\lambda$, $x'=-\lambda$ and $y'=-\beta\lambda$. Moreover, one can easily find that if $\lambda=0$ then $\mu=\infty$.

Now suppose that

$$\lambda=\mu=-1/\lambda \Leftrightarrow \lambda^2=-1.$$

Since -1 is a quadratic non-residue mod q, then $\lambda^2+1=0$ has no solution in $GF(q)$. This gives $\frac{q+1}{2}$ chiral pairs corresponding to the pairs $(W(\lambda), W(\mu))$ of G-invariant submodules.

If $q\equiv 7$ (mod 8) then we have the same equation as in the previous case, i.e. equation (7)

$$\gamma^2+\gamma\beta+\beta^2-\beta+1=0$$

with $x=\frac{1-\gamma-\beta}{\beta}$, $y=\frac{\gamma+1}{\beta}$ and $\beta\neq 0$. Then the discriminant of this quadratic equation is

$$D=-3\beta^2+4\beta-4.$$

Now we claim that there exist δ, $\beta\in GF(q)$ such that

$$\delta^2=D=-3\beta^2+4\beta-4.$$

Indeed, let

$$M=\{\,\delta^2\,|\,\delta\in GF(q)\,\},$$

so $M\subseteq GF(q)$ with $|M|=\frac{q+1}{2}$. Let

$$N=\{-3\beta^2+4\beta-4\,|\,\beta\in GF(q)\}.$$

Since

$$-3\beta^2+4\beta-4=-3(\beta^2-\frac{4}{3}\beta+\frac{4}{3})=-3\left((\beta-\frac{2}{3})^2+\frac{8}{9}\right),$$

and $(\beta-\frac{2}{3})^2$ takes $\frac{q+1}{2}$ different values, so $|N|=\frac{q+1}{2}$. Thus M, $N\subseteq GF(q)$ with

$$|M|+|N|=q+1>|GF(q)|=q,$$

so $M\cap N=\phi$. Hence there exist δ, β such that $\delta^2=-3\beta^2+4\beta-4$, as required. Thus equation (7) has two roots, namely

$$\gamma_1 = \frac{-\beta+\delta}{2},\ \gamma_2 = -\frac{\beta+\delta}{2}.$$

Then we obtain the following two 2-dimensional G-submodules U_1 and U_2 :

$$U_1 = < u_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{-\beta+\delta}{2} \\ \beta \end{pmatrix},\ u_1 = \begin{pmatrix} \frac{-\beta+\delta}{2} \\ \beta \\ -1 \\ 0 \end{pmatrix} >$$

and

$$U_2 = < u_3' = \begin{pmatrix} 1 \\ 0 \\ -\frac{\beta+\delta}{2} \\ \beta \end{pmatrix},\ u_4' = \begin{pmatrix} -\frac{\beta+\delta}{2} \\ \beta \\ -1 \\ 0 \end{pmatrix} >$$

with $H_1 \cong U_1 \oplus U_2$ and $U_1 \cong_G U_2$. The matrix representations of k and l on these submodules are

$$K_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ L_1 = \begin{pmatrix} \frac{-\beta-\delta+2}{2\beta} & \frac{-\beta+\delta+2}{2\beta} \\ \frac{-\beta+\delta+2}{2\beta} & \frac{\beta+\delta-2}{2\beta} \end{pmatrix}$$

and

$$K_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ L_2 = \begin{pmatrix} \frac{-\beta+\delta+2}{2\beta} & \frac{-\beta-\delta+2}{2\beta} \\ \frac{-\beta-\delta+2}{2\beta} & \frac{\beta-\delta-2}{2\beta} \end{pmatrix}$$

respectively.

Now suppose that $\{u_3 = xu_3' + yu_4',\ u_4 = au_3' + bu_4'\}$ is a new basis for U_2 so that k and l induce the matrices $K_1 = K_2$ and $L_1 = L_2$ on U_1 and U_2, i.e

$$Ku_3 = -u_4,\ Ku_4 = u_3$$

and

$$Lu_3 = \frac{-\beta-\delta+2}{2\beta}u_3 + \frac{-\beta+\delta+2}{2\beta}u_4,\ Lu_4 = \frac{-\beta+\delta+2}{2\beta}u_3 + \frac{\beta+\delta-2}{2\beta}u_4.$$

Then $Ku_3 = -u_4$ leads to the following equations:

$$\frac{1}{2}(\beta+\delta)x + y = -a + \frac{1}{2}(\beta+\delta)b,$$

$$\beta x = \beta b,$$

$$x - \frac{1}{2}(\beta+\delta)y = \frac{1}{2}(\beta+\delta)a + b,$$

$$\beta y = -\beta a,$$

and from these equations we obtain $x = b$, $y = -a$. Similarly,

$$Lu_3 = \frac{-\beta-\delta+2}{2\beta}u_3 + \frac{-\beta+\delta+2}{2\beta}u_4$$

yields the following equations:

$$\frac{-\beta-\delta+2}{2\beta}(x - \frac{\beta+\delta}{2}y) + \frac{-\beta+\delta+2}{2\beta}(a - \frac{\beta+\delta}{2}b) = \frac{(2-2\beta)y + (\delta-\beta)x}{2},$$

$$\frac{-\beta-\delta+2}{2\beta}\beta y + \frac{-\beta+\delta+2}{2\beta}\beta b = \frac{(2+\beta-\delta)y + (2-\beta-\delta)x}{2},$$

$$\frac{\beta+\delta-2}{2\beta}(\frac{\beta+\delta}{2}x + y) - \frac{-\beta+\delta+2}{2\beta}(\frac{\beta+\delta}{2}a + b) = \frac{(\delta-\beta)y + (2\beta-2)x}{2},$$

$$\frac{-\beta-\delta+2}{2\beta}\beta x + \frac{-\beta+\delta+2}{2\beta}\beta a = \frac{(2-\beta-\delta)y + (2-\beta+\delta)x}{2},$$

and then putting $x = b$ and $y = -a$ gives

$$b = -\frac{\beta-2}{\delta}a.$$

Now if we choose $a = \delta$, then we get $b = -(\beta-2)$, $x = -(\beta-2)$ and $y = -\delta$. Thus we obtain

$$u_3 = \begin{pmatrix} \beta\frac{2+\delta-3\beta}{2} \\ -\beta\delta \\ \beta\frac{-2+\beta+\delta}{2} \\ -\beta(\beta-2) \end{pmatrix}, \quad u_4 = \begin{pmatrix} \beta\frac{-2+\delta+\beta}{2} \\ -\beta(\beta-2) \\ \beta\frac{-2-\delta+3\beta}{2} \\ \beta\delta \end{pmatrix}.$$

Now defining

$$U(\lambda) = < v_1 = u_1 + \lambda u_3 \ , \ v_2 = u_2 + \lambda u_4 >$$

for any fixed $\lambda \in GF(q)$ and $U(\infty) = U_2$ give $q+1$ different G-isomorphic 2-dimensional G-submodules.

For reflexibility, suppose that there exist $\lambda, \mu \in GF(q)$ such that

$$\alpha_1 U(\lambda) = \alpha_1(<v_1, v_2>) = U(\mu) = <v_1', v_2'>.$$

Then we have

$$\alpha_1 v_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 + \lambda\beta\frac{2+\delta-3\beta}{2} \\ -\lambda\beta\delta \\ \frac{-\beta+\delta}{2} + \lambda\beta\frac{-2+\delta+\beta}{2} \\ \beta - \lambda\beta(\beta-2) \end{pmatrix} = \begin{pmatrix} -(1 + \lambda\beta\frac{2+\delta-3\beta}{2}) \\ \beta - \lambda\beta(\beta-2) \\ \frac{-\beta+\delta}{2} + \lambda\beta\frac{-2+\delta+\beta}{2} \\ -\lambda\beta\delta \end{pmatrix}$$

$$= xv_1' + yv_2' = x\begin{pmatrix} 1+\mu\beta\frac{2+\delta-3\beta}{2} \\ -\mu\beta\delta \\ \frac{-\beta+\delta}{2}+\mu\beta\frac{-2+\delta+\beta}{2} \\ \beta-\mu\beta(\beta-2) \end{pmatrix} + y\begin{pmatrix} \frac{-\beta+\delta}{2}+\mu\beta\frac{-2+\delta+\beta}{2}) \\ \beta-\mu\beta(\beta-2) \\ -1+\mu\beta\frac{-2+\delta+3\beta}{2} \\ \mu\beta\delta \end{pmatrix},$$

for some $x, y \in GF(q)$. This leads to the following equations:

$$\overbrace{(1+\mu\beta\frac{2+\delta-3\beta}{2}}^{A})x + \overbrace{(\frac{-\beta+\delta}{2}+\mu\beta\frac{-2+\delta+\beta}{2}}^{B})y = \overbrace{-(1+\lambda\beta\frac{2+\delta-3\beta}{2}}^{E}),$$

$$\overbrace{(-\mu\beta\delta)}^{C}x + \overbrace{(\beta-\mu\beta(\beta-2))}^{D}y = \overbrace{\beta-\lambda\beta(\beta-2)}^{F},$$

$$\overbrace{(\frac{-\beta+\delta}{2}+\mu\beta\frac{-2+\delta+\beta}{2}}^{B})x - \overbrace{(1+\mu\beta\frac{2+\delta-3\beta}{2}}^{A})y = \overbrace{\frac{-\beta+\delta}{2}+\lambda\beta\frac{-2+\delta+\beta}{2}}^{G},$$

$$\overbrace{(\beta-\mu\beta(\beta-2))}^{D}x - \overbrace{(-\mu\beta\delta)}^{C}y = \overbrace{-\lambda\beta\delta}^{H}.$$

Multiplying the first equation by B and the third by $-A$ and then adding them gives

$$y(A^2+B^2) = BE - AG, \tag{17}$$

and multiplying the second equation by D and the fourth equation by $-C$ and then adding them gives

$$y(C^2+D^2) = DF - CH. \tag{18}$$

Now we claim that

$$A^2+B^2 \neq 0 \text{ and } C^2+D^2 \neq 0.$$

Indeed, the equations

$$\overbrace{(2\beta^3-4\beta^2-2\beta^2\delta)}^{P}\mu^2 + \overbrace{(-4\beta^2)}^{Q} + \overbrace{(2-\beta-\delta)}^{R} = 0$$

and

$$\overbrace{(2\beta^2)}^{P}\mu^2 + \overbrace{(2\beta-4)}^{Q} + \overbrace{-1}^{R} = 0$$

corresponding to $A^2+B^2=0$ and $C^2+D^2=0$, respectively, have discriminants

$$D_1 = Q^2 - 4PR$$

$$\begin{aligned} &= 16\beta^4 - 4(-\beta-\delta+2)(2\beta^3-4\beta^2-2\beta^2\delta) \\ &= 24\beta^4 - 32\beta^3 - 8\delta^2\beta^2 + 32\beta^2 \\ &= -8\beta^2(-3\beta^2+4\beta-4) - 8\delta^2\beta^2 \\ &= -8\beta^2\delta^2 - 8\beta^2\delta^2 \\ &= -16\beta^2\delta^2 \\ &= -1\cdot(16\beta^2\delta^2), \end{aligned}$$

and similarly

$$D_2 = -1\cdot(4\delta^2).$$

Then D_1 and D_2 are quadratic non-residues mod q, (since -1 is a quadratic non-residue mod q). Then equations (17) and (18) give

$$\mu = \frac{2-\beta+\delta-2\lambda\beta^2}{2\beta^2+\lambda(4\beta^2-2\delta\beta^2-2\beta^3)} \quad \text{with} \quad \lambda \neq -\frac{1}{2-\beta-\delta}.$$

It is not difficult to see that if $\lambda = -\dfrac{1}{2-\beta-\delta}$ then $\mu = \infty$, that is,

$$\alpha_1 U(\infty) = \alpha_1(<w_1,\ w_2>) = U(-\frac{1}{2-\beta-\delta}) = <t_1, t_2>,$$

with

$$\alpha_1 w_1 = -\frac{\beta^2+\delta^2}{\delta}t_1 - \frac{(\beta+\delta-2)^2}{2\delta}t_2,$$

and

$$\alpha_1 w_2 = -\frac{(\beta+\delta-2)^2}{2\delta}t_1 + \frac{\delta^2+\beta^2}{\delta}t_2.$$

If $\lambda = 0$ then $\mu = \dfrac{2-\beta+\delta}{2\beta^2}$.

Now suppose that

$$\lambda = \mu = \frac{2-\beta+\delta-2\lambda\beta^2}{2\beta^2+\lambda(4\beta^2-2\delta\beta^2-2\beta^3)},$$

then

$$\overbrace{2\beta^2(2-\delta-\beta)}^{P}\lambda^2 + \overbrace{4\beta^2}^{R}\lambda + \overbrace{\beta-\delta-2}^{S} = 0,$$

so we get a quadratic equation $P\lambda^2 + R\lambda + S = 0$. The discriminant of this equation is

$$D = R^2 - 4PS$$

$$\begin{aligned}
&= 16\beta^4 - 8\beta^2(2-\delta-\beta)(\beta-\delta-2) \\
&= 24\beta^4 - 32\beta^3 - 8\delta^2\beta^2 + 32\beta^2 \\
&= -8\beta^2(-3\beta^2+4\beta-4) - 8\delta^2\beta^2 \\
&= -8\beta^2\delta^2 - 8\beta^2\delta^2 \\
&= -16\beta^2\delta^2 \\
&= -1\cdot(16\beta^2\delta^2).
\end{aligned}$$

Since $16\beta^2\delta^2$ is a quadratic residue mod q and -1 is a quadratic non-residue mod q, then D is a quadratic non-residue mod q. Thus there are $q+1$ q^2-sheeted unbranched regular coverings with genus $\hat{g} = 1+q^2$ and automorphism group

$$\hat{G} \cong \Delta / M_\lambda \cong \mathbb{Z}_q^2 \rtimes Q_8$$

and none is reflexible, i.e. there are $\frac{q+1}{2}$ chiral pairs.

Finally, let $q=2$. Then $\mathbb{Z}_2^4$ is reducible but indecomposable with three 1-dimensional, one 2-dimensional and three 3-dimensional G-invariant submodules, namely

$$S_1 = <\begin{pmatrix}1\\0\\1\\0\end{pmatrix}>,\ S_2 = <\begin{pmatrix}0\\1\\0\\1\end{pmatrix}>,\ S_3 = <\begin{pmatrix}1\\1\\1\\1\end{pmatrix}>,\ S_4 = <\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\end{pmatrix}>,$$

$$S_5 = <\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix}>,\ S_6 = <\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\\1\end{pmatrix}>,$$

$$S_7 = <\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\\1\end{pmatrix}>,$$

respectively.

These submodules correspond to three 8-sheeted regular coverings of genus $\hat{g}=1+q^2=9$ with automorphism group $\hat{G} \cong \mathbb{Z}_2^3 \rtimes Q_8$; one 4-sheeted regular covering of genus $\hat{g}=1+q^2=5$ with automorphism group $\hat{G} = \mathbb{Z}_2^2 \rtimes Q_8$; and three

2-sheeted regular coverings of genus $\hat{g}=1+q^2=3$ with automorphism group $\hat{G}\cong\mathbb{Z}_2\times Q_8$; they are all reflexible hypermaps.

(II). Let $\Delta=\Delta(3,3,4)=<x,y,z \mid x^3=y^3=z^4=xyz=1>$ and $G=SL_2(3)=<d,e \mid d^3=e^4=1,\ ed^2e=ded>$. Then the homology representation and character are

$$\rho_1: d\mapsto\begin{pmatrix}0&1&0&0\\-1&-1&0&0\\1&0&-1&-1\\0&1&1&0\end{pmatrix}=D,\ e\mapsto\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}=E$$

and

$$\tau_1=2\chi_5,$$

where χ_5 is an irredicuble complex character of G given by $\chi_5(1)=2$, $\chi_5(e^2)=-2,\ \chi_5(e)=0,\ \chi_5(d)=-1,\ \chi_5(d^2)=-1,\ \chi_5(d^2e^2)=1,\ \chi_5(de^2)=1$ (where $1,e^2,e,d,d^2,d^2e^2,de^2$ are the representatives of conjugacy classes of G) [11].

Since $Q_8\trianglelefteq SL_2(3)$, all $SL_2(3)$-invariant submodules must be Q_8-invariant as well, so it is enough to check which Q_8-invariant submodules are also $SL_2(3)$-invariant (where $Q_8=<e,d^{-1}ed>$ as in (I)).

If $q\equiv 1,5\ (\mathrm{mod}\ 8)$ then all Q_8-invariant submodules are also $SL_2(3)$-invariant, so there are $q+1$ q^2-sheeted regular unbranched coverings of type $\{3,3,4\}$ with genus $1+q^2$ and automorphism group $\hat{G}=\Delta/M_\lambda\cong\mathbb{Z}_q^2\rtimes SL_2(3)$ for all $\lambda\in GF(q)\cup\{\infty\}$. On the other hand, it was shown in [11] that the matrix of the orientation–reversing automorphism corresponding to the chosen reflection line and the homology basis is the same as in (I), that is,

$$\alpha_1=\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

Now suppose that there exist $\lambda,\mu\in GF(q)\cup\{\infty\}$ such that

$$\alpha_1U(\lambda)=U(\mu)=<v_1',v_2'>$$

then it follows that

$$\alpha_1 v_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1-\lambda \\ \lambda(1-i) \\ i(\lambda-1) \\ -\lambda(i+1) \end{pmatrix} = \begin{pmatrix} \lambda(i-1) \\ \lambda-1 \\ -\lambda(i+1) \\ i(\lambda-1) \end{pmatrix}$$

$$= xv_1' + yv_2' = x\begin{pmatrix} 1-\mu \\ \mu(1-i) \\ i(\mu-1) \\ -\mu(i+1) \end{pmatrix} + y\begin{pmatrix} i-1 \\ 1+\mu \\ -\mu(i+1) \\ \mu+i \end{pmatrix},$$

for some $x, y \in GF(q)$. This yield the following equations:

$$x(1-\mu) + y(i-1) = \lambda(i-1), \tag{19}$$

$$x\mu(1-i) + y(1+\mu i) = \lambda - 1, \tag{20}$$

$$xi(\mu-1) - y\mu(i+1) = -\lambda(i+1), \tag{21}$$

$$-x\mu(i+1) + y(\mu+1) = i(\lambda-1). \tag{22}$$

Multiplying equation (20) by i and adding it to equation (22) gives

$$y = \lambda - 1.$$

Similarly, multiplying equation (19) by i and adding it to equation (21) gives

$$y = \frac{2\lambda}{\mu+1}.$$

Then equating these two values of y gives $\mu = \dfrac{\lambda+1}{\lambda-1}$, $\lambda \neq 1$.

For $\lambda = 1$ we see that $\mu = \infty$, that is, $\alpha_1 W(1) = W(\infty)$.

Now suppose that there is a proper reflexible hypermap, that is, $\lambda = \mu = \dfrac{\lambda+1}{\lambda-1}$ then this gives

$$\lambda^2 - 2\lambda - 1 = 0.$$

This equation has solutions $\lambda_{1,2} = 1 \pm \xi$ (where $\xi^2 = 2$) if and only if $q \equiv 1 \pmod 8$. Thus there are two reflexible hypermap corresponding to the submodules $U(1+\xi)$ and $U(1-\xi)$ and $\dfrac{q-1}{2}$ chiral pairs corresponding to the pairs of submodules $(U(\lambda),\ U(\dfrac{\lambda+1}{\lambda-1}))$ (where $\lambda \neq 1$).

If $q \equiv 5 \pmod 8$ then 2 is a quadratic non-residue mod q, so there are no proper reflexible hypermap, i.e. there are $\dfrac{q+1}{2}$ chiral pairs.

If $q \equiv 3 \pmod 8$, then one can easily see that all Q_8-invariant submodules are also $SL_2(3)$-invariant submodules, so there are $q+1$ 2-dimensional G-invariant submodules. It can be shown that $\frac{q-1}{2}$ of them are chiral pairs corresponding to $(U(\lambda), U(\frac{1}{\lambda}))$ with $\lambda \neq 0, \pm 1$ and $(U(0), U(\infty))$; and two of them are reflexible hypermaps corresponding to $U(1)$ and $U(-1)$.

Now let $q \equiv 7 \pmod 8$. Then all Q_8-invariant submodules are also $SL_2(3)$-invariant, so there are $q+1$ q^2-sheeted unbranched regular coverings. They are all chiral. Indeed, for any $\lambda, \mu \in GF(q)$,

$\alpha_1 U(\lambda) = U(\mu)$ if and only if $\mu = \dfrac{2-\delta-3\beta-\lambda(4\beta)}{4\beta+\lambda(-6\beta^3+4\beta^2+2\delta\beta^2)}$,

with

$$\lambda \neq -\frac{2}{-3\beta^2+2\beta+\delta\beta} \text{ and } \delta^2 = -3\beta^2+4\beta-4.$$

For

$$\lambda = -\frac{2}{-3\beta^2+2\beta+\delta\beta},$$

one can see that $\mu = \infty$.

Now suppose that $\lambda = \mu$, then we obtain a quadratic equation

$$\overbrace{2\beta^2(-2-\delta+3\beta)}^{P}\lambda^2 + \overbrace{-8\beta}^{R}\lambda + \overbrace{(2-\delta-3\beta)}^{S} = 0,$$

i.e. $P\lambda^2 + R\lambda + S = 0$. Then the discriminant of this equation is

$$\begin{aligned}
\mathrm{D} &= R^2 - 4PS \\
&= 64\beta^2 + 8\beta^2(-2+\delta+3\beta)(-2-\delta+3\beta) \\
&= 64\beta^2 + 8\beta^2(9\beta^2-12\beta+4-\delta^2) \\
&= 64\beta^2 + 8\beta^2\left[3(\beta^2-4\beta+4)-8-\delta^2)\right] \\
&= 64\beta^2 + 8\beta^2(-3\delta^2-\delta^2-8) \\
&= 64\beta^2 - 32\beta^2\delta^2 - 64\beta^2 \\
&= -32\beta^2\delta^2 \\
&= -2\cdot(16\beta^2\delta^2).
\end{aligned}$$

Now $16\beta^2\delta^2$ is a quadratic residue mod q, but -2 is a quadratic non-residue mod q, so $-2\cdot(16\beta^2\delta^2)$ is a quadratic non-residue mod q. Thus there are no proper reflexible hypermaps, so there are $\frac{q+1}{2}$ chiral pairs corresponding to the pairs of submodules $(U(\lambda),U(\mu))$.

If $q=2$, then there is only one Q_8 – invariant submodule,

$$S_4 = < \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} >,$$

which is also an $SL_2(3)$ – invariant submodule. Therefore, there is only one 4 – sheeted unbranched covering with genus 5 and $\hat{G} \cong \mathbb{Z}_2^2 \rtimes SL_2(3)$.

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