Commun. Fac. Sci. Univ. Ank. Series A1
V. 49. pp. 15 - 23 (2000)

ON HARMONIC CURVATURES OF A FRENET CURVE

K. ARSLAN[1], Y. ÇELİK[2] and H.H. HACISALİHOĞLU[3]

[1]*Uludağ University, Faculty of Science, Dept. of Mathematics, Bursa, Turkey.*
[2]*DumlupınarUniversity, Faculty of Science, Dept. of Mathematics, Kütahya, Turkey.*
[3]*Ankara University, Faculty of Science, Dept. of Mathematics, 06100, Tandoğan, Ankara, Turkey.*

(Received Oct. 14, 1999; Revised Feb. 9, 2000; Accepted Feb. 16, 2000)

ABSTRACT

In this paper we consider harmonic curvatures of a Frenet curve of osculating order d. We also consider a general helix of AW(k) type where k=1,2 and 3. We have show that, there is no general helix of AW(3) type. We also show that, a general helix of rank 1 and of AW(3) type must be a circular helix. We give curvature conditions of the curves, which have AW(2) property. In the final part, we visualize the plane and space curves of (weak) AW(2) type.

1. INTRODUCTION

In this part we consider Frenet curves (of osculating order d) and their harmonic curvatures.

Let $\gamma : I \subseteq \mathbf{E} \to \mathbf{E}^n$ be a unit speed curve in $\mathbf{E}^n$. The curve γ is called Frenet curve of osculating order d if its higher order derivatives $\gamma'(s), \gamma''(s), \gamma'''(s), \ldots, \gamma^{(d)}(s)$ are linearly independent and $\gamma'(s), \gamma''(s), \gamma'''(s), \ldots, \gamma^{(d+1)}(s)$ are no longer linearly independent for all $s \in I$. For each Frenet curve of order d one can associate an orthonormal d-frame $v_1, v_2, \ldots, v_d$ along γ (such that $\gamma'(s) = v_1$) called the Frenet frame and $d-1$ functions $\kappa_1, \kappa_2, \ldots, \kappa_{d-1} : I \to \mathbf{R}$, called the Frenet curvatures, such that the Frenet formulas are defined in the usual way;

$$D_{v_1}\gamma'(s) = \kappa_1(s)v_2(s), \tag{1}$$

$$D_{v_1}v_2(s) = -\kappa_1(s)\gamma'(s) + \kappa_2(s)v_3(s), \tag{2}$$

....

$$D_{v_1}v_i(s) = -\kappa_{i-1}(s)v_{i-1}(s) + \kappa_i(s)v_{i+1}(s), \tag{3}$$

$$D_{v_1}v_{i+1}(s) = -\kappa_i(s)v_i(s) \tag{4}$$

where D is the Levi-civita connection of $\mathbf{E}^n$.

A regular curve $\gamma : I \subseteq \mathbf{E} \to \mathbf{E}^n$ is called a W-curve of rank d, if γ is a Frenet curve of osculating order d and the Frenet curvatures κ_i, $1 \le i \le d-1$ are non-zero constants. A W-curve of rank 3 is a right circular helix [4].

Let γ be a unit speed curve of osculating order d. The functions $H_j : I \to \mathbf{E}$, $1 \le j \le d-2$ defined by

$$H_1 = \frac{\kappa_1}{\kappa_2}, H_j = \{D_{v_1} H_{j-1} + H_{j-2}\kappa_j\}\frac{1}{\kappa_{j+1}}, \quad 2 \le j \le d-2 \tag{5}$$

are called the harmonic curvatures of γ where $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$ are Frenet curvatures of γ which are not necessarily constant.

Definition 1. The unit speed Frenet curve of osculating order d is called general helix of rank $(d-2)$ if $\sum_{i=1}^{d-2} H_i^2 = c$, where c is any constant.

2. Frenet Curve of Osculating Order 3

In this part, we consider the Frenet curve of osculating order 3 of $\mathbf{E}^n$. We obtain some results. First, we start with some well known results;

Proposition 2. Let γ be a Frenet curve in $\mathbf{E}^n$ of osculating order 3 then

$$D_{v_1} v_1(s) = \kappa_1(s) v_2(s), \tag{6}$$

$$D_{v_1} v_2(s) = -\kappa_1(s) v_1(s) + \frac{\kappa_1(s)}{H_1(s)} v_3(s), \tag{7}$$

$$D_{v_1} v_3(s) = -\frac{\kappa_1(s)}{H_1(s)} v_2(s), \tag{8}$$

where H_1 is the first harmonic curvature of γ and $\gamma'(s) = v_1, v_2, v_3$ are the Frenet frame fields.

Proof. Using the Frenet formulas (1)-(4) and the equation (5), we get the result.

Theorem 3. [6] The higher order derivatives $\gamma''(s), \gamma'''(s), \gamma''''(s)$ are linearly dependent if and only if γ is a general helix of rank 1.

Remark 1. By previous proposition, it is easy to show that a general helix of rank 1 is of order 3.

Proposition 4 [6]. Let γ be a Frenet curve of $\mathbf{E}^n$ of osculating order 3. Then γ is a general helix of rank 1 if and only if H_1 = constant,

Theorem 5 [6]. Let γ be a general helix of rank 1. For the Frenet frame v_1, v_2, v_3, the harmonic curvature of γ is given by $H_1 = \dfrac{\langle v_3, X\rangle}{\langle v_1, X\rangle}$, where X is the fixed unit vector.

Proposition 6. Let γ be a Frenet curve in $\mathbf{E}^n$ of osculating order 3 then we have

$$\gamma''(s) = D_{v_1}\gamma'(s) = \kappa_1(s)v_2, \tag{9}$$

$$\gamma'''(s) = D_{v_1}D_{v_1}\gamma'(s) = -\kappa_1^2(s)v_1 + \kappa_1'(s)v_2 + \frac{\kappa_1^2(s)}{H_1(s)}v_3,$$

$$\gamma''''(s) = D_{v_1}D_{v_1}D_{v_1}\gamma'(s) = -3\kappa_1(s)\kappa_1'(s)v_1 + \{\kappa_1''(s) - \kappa_1^3(s) - \frac{\kappa_1^3(s)}{H_1^2(s)}\}v_2 + \{\frac{3\kappa_1(s)\kappa_1'(s)H_1(s) - \kappa_1^2 H_1'(s)}{H_1^2(s)}\}v_3,$$

where $\gamma'(s) = v_1$ and H_1 is the first harmonic curvature.

Proof. Suppose $\gamma : I \subseteq \mathbf{E} \to \mathbf{E}^n$ is a unit speed curve in $\mathbf{E}^n$. If γ is a Frenet curve of order 3 then by the use of equations (6)-(8), we get the result.

Notation. Let us write

$$N_1(s) = \kappa_1(s)v_2, \tag{10}$$

$$N_2(s) = \kappa_1'(s)v_2 + \frac{\kappa_1^2(s)}{H_1(s)}v_3, \tag{11}$$

$$N_3(s) = \{\kappa_1''(s) - (\kappa_1(s))^3(1 + \frac{1}{H_1^2(s)}\}v_2 + \{\frac{3\kappa_1(s)\kappa_1'(s)H_1(s) - \kappa_1^2 H_1'(s)}{H_1^2(s)}\}v_3. \tag{12}$$

Proposition 7 [1]. Let γ be a Frenet curve in $\mathbf{E}^n$ of osculating order 3 then

$$\begin{aligned} &\{\|N_1\|^2\|N_2\|^2 - \langle N_1, N_2\rangle^2\}N_3 \\ &= \{\|N_2\|^2\langle N_3, N_1\rangle - \langle N_3, N_2\rangle\langle N_1, N_2\rangle\}N_1 \\ &+ \{\|N_1\|^2\langle N_3, N_2\rangle - \langle N_3, N_1\rangle\langle N_1, N_2\rangle\}N_2 \end{aligned}$$

By the use of above proposition we obtain;

Theorem 8. Let γ be a Frenet curve in $\mathbf{E}^n$ of osculating order 3 then

$$N_3(s) = \langle N_3(s), N_1^*(s)\rangle N_1^*(s) + \langle N_3(s), N_2^*(s)\rangle N_2^*(s)$$

where

$$N_1^*(s) = \frac{N_1(s)}{\|N_1(s)\|}$$

and

$$N_2^*(s) = \frac{N_2(s) - \langle N_2(s), N_1^*(s)\rangle N_1^*(s)}{\left\| N_2(s) - \langle N_2(s), N_1^*(s)\rangle N_1^*(s)\right\|}.$$

First we will give the definition of the curve of AW(k) type (see [1]).

Definition 9. Frenet curves (of osculating order 3) are

i) of type AW(1) if they satisfy $N_3(s)=0$,

ii) of type AW(2) if they satisfy

$$\|N_2(s)\|^2 N_3(s) = \langle N_3(s), N_2(s)\rangle N_2(s), \tag{13}$$

iii) of type AW(3) if they satisfy

$$\|N_1(s)\|^2 N_3(s) = \langle N_3(s), N_1(s)\rangle N_1(s). \tag{14}$$

In the followings, we consider Frenet curves of order 3 to be of type AW(k), where k=1,2 and 3.

Corollary 10. Let γ be a Frenet curve of osculating order 3 then $N_1^* = \upsilon_2$, $N_2^* = \upsilon_3$.

Definition 11. Frenet curves (of osculating order 3) are

i) of type weak AW(2) if they satisfy

$$N_3(s) = \langle N_3(s), N_2^*(s)\rangle N_2^*(s), \tag{15}$$

ii) of type weak AW(3) if they satisfy

$$N_3(s) = \langle N_3(s), N_1^*(s)\rangle N_1^*(s), \tag{16}$$

where $N_1^* = \upsilon_2$, $N_2^* = \upsilon_3$.

By the use of Proposition 6 and Definition 11, we obtain the following results;

Corollary 12. Every Frenet curve of weak AW(3) type is of AW(3) type.

Corollary 13. Let γ be a Frenet curve of order 3. If γ is of type weak AW(2) then

$$\kappa_1''(s) - \frac{\kappa_1^2(s)}{H_1^2} - \kappa_1^3(s) = 0, \tag{17}$$

where $H_1 = \frac{\kappa_1}{\kappa_2}$ is the first harmonic curvature of γ.

Corollary 14. Let γ be a Frenet curve of type weak AW(2). If γ is a plane curve then $\kappa_1''(s)-\kappa_1^3(s)=0$, and the solution of this differential equation is $\kappa_1(s)=\pm\frac{\sqrt{2}}{s+c}$, where c is constant (see Figure 1).

Theorem 15. Let γ be a general helix of rank 1. If γ is of type weak AW(2) then

$$\kappa_1(s)=\pm\frac{\sqrt{2}}{\sqrt{A\,s+c}},\ \text{and}\ \kappa_2(s)=\sqrt{A-1}\,\kappa_1(s)$$

where $A=1+\frac{1}{H_1^2(s)}$ (see Figure 2).

Proof. Suppose that γ is a Frenet curve of type weak AW(2) then its Frenet curvatures κ_1,κ_2 satisfiy the differential equation (17). If we solve this equation we get the result.

Theorem 16. Let γ be a Frenet curve of order 3. Then there is no (circular or general) helix of type AW(1).

Proof. Assume that γ be a helix then by definition H_1=constant. So $H_1'=0$. Therefore the system of the differential equation

$$3\kappa_1(s)\kappa_1'(s)H_1(s)=0, \tag{18}$$

$$\kappa_1''(s)-\kappa_1(s)^3(1+\frac{1}{H_1^2(s)})=0, \tag{19}$$

does not have a non-trivial solution.

Theorem 17. Let γ be a Frenet curve of order 3. Then γ is of type AW(2) if and only if

$$3(\kappa_1'(s))^2\kappa_1(s)H_1(s)-\kappa_1^2(s)\kappa_1'(s)H_1'(s)-\kappa_1^2(s)\kappa_1''(s)+\kappa_1^5(s)H_1(s)(1+\frac{1}{H_1^2(s)})=0. \tag{20}$$

Proof. Suppose γ is a Frenet curve of order 3 then by (11) and (12), we can write

$$N_2(s)=\alpha(s)Y_2+\beta(s)Y_3, \tag{21}$$

$$N_3(s)=\eta(s)Y_2+\delta(s)Y_3, \tag{22}$$

where α,β,η and δ are differentiable functions. Since γ is of type AW(2). Then by Definition 9, the vectors $N_2(s)$ and $N_3(s)$ are linear depended. So

$$\begin{vmatrix}\alpha & \beta\\ \eta & \delta\end{vmatrix}=0;\ \text{i.e.}\ \alpha(s)\delta(s)=\beta(s)\eta(s). \tag{23}$$

Using (10)-(12) we get

$$\alpha(s)=\kappa_1'(s),\ \beta(s)=\frac{\kappa_1^2(s)}{H_1(s)},$$

$$\eta(s)=\kappa_1''(s)-(\kappa_1(s))^3(1+\frac{1}{H_1^2(s)}),$$

$$\delta(s)=\frac{3\kappa_1(s)\kappa_1'(s)H_1(s)-\kappa_1^2(s)H_1'(s)}{H_1^2(s)}.$$

Substituting these into (23), we obtain (20).

Conversely if the equation (20) holds it is easy to show that γ is of AW(2) type. This completes the proof.

Corollary 18. Let γ be a Frenet curve of order 3. If γ is general helix of type AW(2) then, it satisfies

$$3(\kappa_1'(s))^2-\kappa_1(s)\kappa_1''(s)+\kappa_1^4(s)(1+\frac{1}{H_1^2(s)})=0. \tag{24}$$

Theorem 19. Let γ be a Frenet curve of order 3. Then it is of type AW(3) if and only if

$$3\kappa_1'(s)H_1(s)-\kappa_1^2(s)H_1'(s)=0. \tag{25}$$

Proof. Suppose γ is a Frenet curve of order 3 which is of type AW(3). So substituting (10) and (12) into (16) we get (25).

Conversely if the equation (25) holds, it is easy to show that γ is of AW(3) type. This completes the proof of the theorem.

Corollary 20. Let γ be a Frenet curve of order 3 and of type AW(3). If γ is a general helix of rank 1 then it must be a circular helix.

Proof. Suppose γ is a general helix of rank 1, then by definition $H_1'(s)=0$. So the equation (24) becomes $\kappa_1'(s)H_1(s)=0$. Since γ is a space curve, $H_1(s)$ is none zero so $\kappa_1'(s)=0$ (i.e. κ_1 is constant). By the definition of general helix κ_2 must be constant too. So γ must be a circular helix.

Theorem 21. Let γ be a general helix of rank 1. If γ is of AW(2) type then

$$\kappa_1(s)=\frac{1}{\sqrt{-As^2+Bs+C}} \text{ and } \kappa_2(s)=\sqrt{A-1}\ \kappa_1(s), \tag{26}$$

where $A=1+\frac{1}{H_1^2(s)}$ B and C are real constants and H_1 is the first harmonic curvature of γ (see Figure 3).

Proof. Suppose γ is a general helix of AW(2) type. If we substitute $\kappa_1 = x$ in (24) we get

$$x\frac{d^2x}{ds^2} - 3(\frac{dx}{ds})^2 = Ax^4, \ A = 1 + \frac{1}{H_1^2(s)}. \tag{27}$$

Let us take $x = y^p$ and differentiating it twice $\frac{dx}{ds} = py^{p-1}\frac{dy}{ds}$, and $\frac{d^2x}{ds^2} = p(p-1)y^{p-2}(\frac{dy}{ds})^2 + py^{p-1}\frac{d^2y}{ds^2}$ so the equation (27) becomes

$$y^p\left[py^{p-1}\frac{d^2y}{ds^2} + p(p-1)y^{p-2}(\frac{dy}{ds})^2\right] - 3p^2y^{2p-2}(\frac{dy}{ds})^2 = Ay^{4p}, \tag{28}$$

$$py^{2p-1}\frac{d^2y}{ds^2} + p(p-1)y^{2p-2}(\frac{dy}{ds})^2 - 3p^2y^{2p-2}(\frac{dy}{ds})^2 = Ay^{4p}, \tag{29}$$

Putting $p(p-1)=3p^2$ (i.e. $p = -\frac{1}{2}$) into the last equation we get

$$py^{2p-1}\frac{d^2y}{ds^2} = Ay^{4p}, \ y^{-2}(-\frac{1}{2}\frac{d^2y}{ds^2}) = A. \tag{30}$$

So $\frac{d^2y}{ds^2} = -2A$. Now, we solve this equation. Since $\frac{dy}{ds} = -2At + B$, we get $y = -As^2 + Bs + C$. By the use of $x = y^{-\frac{1}{2}}$ we obtain $x = (-As^2 + Bs + C)^{-\frac{1}{2}}$. Since $H_1 = \frac{\kappa_1}{\kappa_2}$, we have the result.

Corollary 22. Let γ be a Frenet curve of osculating order 3. If γ is of AW(2) type then γ can not be a circular helix.

Proof. Let γ be a Frenet curve of order 3. If γ is of type AW(2) then by Theorem 8 we get

$$3(\kappa_1'(s))^2\kappa_1(s)H_1(s) - \kappa_1^2(s)\kappa_1'(s)H_1'(s) - \kappa_1^2(s)\kappa_1''(s) + \kappa_1^5(s)H_1(s)(1 + \frac{1}{H_1^2(s)}) = 0.$$

Assume that γ is circular helix then $\kappa_1(s)$ must be non zero constant. So we get $\kappa_1^5(s)H_1(s)(1 + \frac{1}{H_1^2(s)}) = 0$. Since γ can not be a straight line then $(1 + \frac{1}{H_1^2(s)} = 0$, which is impossible. So γ can not be circular helix.

3. Visualization

We visualize the space curves mentioned in the previous section making use of Mathematica. Here, we show just a simple command to plot a portion of a space curve. In the mathematica session below kk and tt denote the first and second Frenet curvature of a space curve. Using kk and tt values, it is possible to plot a portion of a plane curve (kk=0) and a space curve. To do this, we use the following programme (see [5]);

```
plotintrinsic3d[{kk_,tt_},{a_ : 0, {p1_ : 0, p2_ : 0, p3_ : 0},
{q1_:0, q2_:0, q3_:0}, {p1_ : 0, p2_ : 0, p3_ : 0}, {smin_10,smax:10},opts_]:=
ParametricPlot3D[Module[{x1,x2,x3,t1,t2,t3,n1,n2,n3,b1,b2,b3},
{x1[s],x2[s],x3[s]}/.
NDSolve[{x1'[ss]==t1[ss], x2'[ss]==t2[ss], x3'[ss]==t3[ss],
t1'[ss]==kk[ss]n1[ss], t2'[ss]==kk[ss]n2[ss], t3'[ss]== kk[ss]n1[ss],
n1'[ss]==-kk[ss]t1[ss]+tt[ss]b1[ss],
n2'[ss]==-kk[ss]t2[ss]+tt[ss]b2[ss],
n3'[ss]==-kk[ss]t3[ss]+tt[ss]b3[ss],
b1'[ss]==-tt[ss]n1[ss], b2'[ss]==-tt[ss]n2[ss], b3'[ss]==-tt[ss]n3[ss],
x1[a]==p1, x2[a]==p2 , x3[a]==p3,
t1[a]==q1, t2[a]==q2, t3[a]==q3,
n1[a]==r1, n2[a]==r2, n3[a]==r3,
b1[a]==q2r3-q3r2, b2[a]==q3r1-q1r3, b3[a]==q1r2-q2r1,
{x1,x2,x3,t1,t2,t3,n1,n2,n3,b1,b2,b3},
{ss,smin,smax}]]//Evaluate, {s,smin,smax},opts];
```

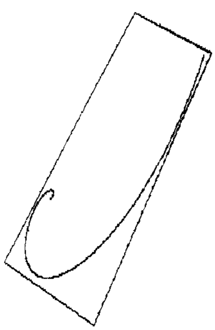
Figure 1

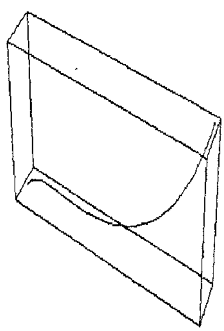
Figure 2

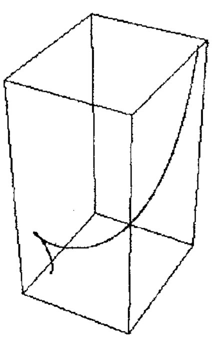
Figure 3

REFERENCES

[1] Arslan, K. and West, A., Product Submanifolds with Pointwise 3-Planar Normal Sections, Glasgow Math. J. 37,(1995), 73-81.

[2] Dillen F., The Classification of Hypersurfaces of a Euclidean Space with Parallel Higher Order Fundamental Form.}} Math Z. 203(1990), 635-643.

[3] Dillen, F. and Nölker, S., Semi-parallelity, multy-rotation surfaces and the helix-property, J. Reine Angew. Math. 435,(1993), 33-63.

[4] Ferus, D. and Schirrmacher, S., Submanifolds in Euclidean Space with Simple Geodesics, Math. Ann. 260 (1982), 57-62.

[5] Gray, Alfred., Modern Differential Geometry of Curves and Surfaces, CRS Press, 1993.

[6] Hacisalihoğlu, H.H., Diferensiyel Geometri, Ankara Üniversitesi Fen Fakültesi, 1993.

[7] Ross, S.L., Differential Equations, Second Edition, John Wiley&Sons Inc., 1974.