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SUMMABILITY OF TRIGONOMETRIC SEQUENCES BY SEQUENCE OF INFINITE MATRICES

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ABSTRACT

Using the concept of almost-regular matrices (introduced by King [3]), Rao [6] proved a result for the A-summability of the derived series of a Fourier series. Mursaleen [5] generalized the result of Rao using the concept of σ-regular matrices and also proved corresponding result for conjugate series. The purpose of this paper is to generalize these results by using $\mathcal{B}$ (or $F_{\mathcal{B}}$) -regular matrices which is the generalization of almost and σ-regular matrices.

1. DEFINITIONS AND NOTATIONS

Let $f(x)$ be a periodic function, with period 2π, and integrable (L), that is, integrable in the sense of Lebesgue over $(-\pi,\pi)$. Let the Fourier series of $f(x)$ be given by

$$\frac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx). \tag{1.1}$$

Then the series conjugate to it is

$$\sum_{k=1}^{\infty}(b_k\cos kx-a_k\sin kx) \tag{1.2}$$

and its derived series is

$$\sum_{k=1}^{\infty}k(b_k\cos kx-a_k\sin kx). \tag{1.3}$$

Let $\tilde{s}_n(x)$ and $s'_n(x)$ denote the partial sums of the series (1.2) and (1.3) respectively.

We write

(1.4) $$\psi_x(t)=\psi(f,t)=\begin{cases} f(x+t)-f(x-t), & 0<t\le\pi, \\ g(x) & , \quad t=0 \quad , \end{cases}$$

where

(1.5) $$g(x)=\{f(x+0)-f(x-0)\};$$

and

(1.6) $$h_x(t)=\frac{\psi_x(t)}{4\sin\frac{t}{2}}.$$

Let $\mathcal{B}=(B_i)$, with $B_i=(b_{nk}(i))$, be a sequence of infinite matrices. Then, a sequence $x=(x_n)\in\ell_\infty$ is said to be $\mathcal{B}$ -(or $F_\mathcal{B}$) -convergent or $\mathcal{B}$ summable to the generalized limit $\mathcal{B}_x$, if

(1.7) $$\lim_{n\to\infty}(B_ix)_n=\lim_{n\to\infty}\sum_{k=0}^{\infty}b_{nk}(i)x_k=Lim\ \mathcal{B}_x,$$

uniformly for $i=0,1,2,\ldots$.

The space $\mathcal{B}$ (or $F_\mathcal{B}$) of $\mathcal{B}$ -convergent (summable) sequences are thus defined as (see Stielglitz [7])

(1.8) $$c^\mathcal{B}:=\{x\in\ell_\infty:\lim t_{in}^\mathcal{B}(x)=L, \quad \text{uniformly in } i\},$$

where

$$t_{in}^\mathcal{B}(x)=\sum_{k=0}^{\infty}b_{nk}(i)x_k,$$

and

$$L=\text{Lim}\,\mathcal{B}x.$$

For an infinite matrix $A=(a_{nk})$, we write

$$Ax=((Ax)_n), \quad with\ (Ax)_n=\sum_k a_{nk}x_k.$$

In the case in which $\mathcal{B}_0=(I)$, the identity matrix, $\mathcal{B}$ reduces to c, the space of convergent sequences, and for $\hat{\mathcal{B}}=(\hat{B}_i)$ and $\mathcal{B}^\sigma=(B_i^\sigma)$, $\mathcal{B}$ becomes the spaces $\hat{c}$ of almost convergent sequences and c^σ of σ-convergent sequences respectively, where

(1.9) $$\hat{b}_{nk}(i)=\begin{cases}(n+1)^{-1} & , \quad \text{for } i\le k\le i+n, \\ 0 & , \quad \text{elsewhere}\end{cases}$$

and

(1.10) $$b_{nk}^\sigma(i)=\begin{cases} n^{-1} & , \quad \text{for } \sigma^1(i)\le k\le\sigma^n(i) \\ 0 & , \quad \text{elsewhere}\end{cases}$$

Let us consider the transformation

$$\mathcal{B}(Ax) = (B_i(Ax))_n$$
$$= \sum_{k=0}^{\infty} b_{nk}(i) A_k(x)$$
(1.11)
$$= \sum_{k=0}^{\infty} b_{nk}(i) \sum_{l=0}^{\infty} a_{kl} x_l$$
$$= \sum_{l=0}^{\infty} (\sum_{k=0}^{\infty} b_{nk}(i) a_{kl}) x_l .$$

Then, using (1.9) and (1.10), (1.11) becomes

$$\hat{\mathcal{B}}(Ax) = (B_i(Ax))_n$$
$$= \sum_{l=0}^{\infty} (\tfrac{1}{n+1} \sum_{k=i}^{n+i} a_{kl}) x_l$$
$$= \tfrac{1}{n+1} \sum_{j=0}^{n} \sum_{l=0}^{\infty} a_{j+i,l} x_l$$
$$= \tfrac{1}{n+1} \sum_{j=0}^{n} \sum_{l=0}^{\infty} a(j+i,l) x_l ,$$

and

$$\mathcal{B}^{\sigma}(Ax) = (B_i^{\sigma}(Ax))_n$$
$$= \sum_{l=0}^{\infty} \tfrac{1}{n} \sum_{k=\sigma^1(i)}^{\sigma^n(i)} a_{kl} x_l$$
$$= \sum_{l=1}^{\infty} \tfrac{1}{n} \sum_{j=1}^{n} a(\sigma^j(n), l) x_l$$
$$= \tfrac{1}{n} \sum_{j=1}^{n} \sum_{l=1}^{\infty} a(\sigma^j(n), l) x_l$$

where, for the matrix $A = (a_{nk})$, we use the notation

$$a_{nk} = a(n,k).$$

2. INTRODUCTION

Using the concept of almost-regular matrices (introduced by King [3]), Rao [6] proved the following results for the A-summability of the derived series of a Fourier series.

Theorem 2.1. Let $A = (a_{nk})$ be an almost regular infinite matrix of real numbers. Then, for every $x \in [-\pi, \pi]$ for which $h_x(t) \in BV[0, \pi]$,

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \sum_{l=0}^{\infty} a_{j+i,l} s_l'(x) = h_x(0+),$$

uniformly in i, if and only if

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \sum_{l=0}^{\infty} a_{j+i,l} \sin(l+\frac{1}{2})t = 0,$$

for every $t \in [0,\pi]$, uniformly in i.

Generalizing the above result using the concept of σ – regular matrices (which is the generalization of almost regular matrices) Mursaleen [5] proved the following result.

Theorem 2.2. ([5], Chapter VI, Theorem 1). Let $A = (a_{nk})$ be a σ – regular matrix. Then, for each $x \in [-\pi,\pi]$ for which $h_x(t) \in BV[0,\pi]$,

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} \sum_{l=1}^{\infty} a(\sigma^j(i),l) \; s_l'(x) = h_x(0+),$$

uniformly in i, if and only if

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} \sum_{l=1}^{\infty} a(\sigma^j(i),l) \; \sin(l+\frac{1}{2})t = 0,$$

for every $t \in [0,\pi]$, uniformly in i.

Mursaleen [5] also proved corresponding result for conjugate series.

Theorem 2.3. ([5], Chapter VI, Theorem 2). Let $A = (a_{nk})$ be a σ – regular matrix. Then for each $x \in [0,2\pi]$ for which $f(x) \in BV[0,2\pi]$,

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{\infty} a(\sigma^j(i),k) \; \tilde{s}_k(x) = \pi^{-1} g(x),$$

uniformly in i, if and only if

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{\infty} a(\sigma^j(i),k) \; \cos kt = 0,$$

for all $t \in [\delta,\pi]$, $\delta > 0$, uniformly in n.

The purpose of the present paper is to generalize above-cited results by using $\mathcal{B}$ (or $F_{\mathcal{B}}$)-regular matrices (see Stieglitz [7], Satz 3(b)) which is the generalization of almost and σ -regular matrices.

3. MAIN RESULTS

We establish the following theorems.

Theorem 3.1. Let $\mathcal{B} = (B_i)$ be a family of matrices, with

(3.1) $$N(B_i) < \infty, \quad \text{for each } i.$$

Let $A = (a_{nk})$ be a $\mathcal{B}$ (or $F_{\mathcal{B}}$)-regular matrix, i.e. $A \in (c,\mathcal{B})_{reg}$. Then for every $x \in [-\pi,\pi]$ for which $h_x(t) \in BV[0,\pi]$,

(3.2) $$\lim_n \sum_l \sum_k b_{nk}(i) a_{kl} s_l'(x) = h_x(0+),$$

uniformly in i, if and only if

(3.3) $$\lim_n \sum_l \sum_k^{\infty} b_{nk}(i) a_{kl} \sin(l+\frac{1}{2})t = 0.$$

Theorem 3.2. Let $\mathcal{B} = (B_i)$ be a family of matrices, with the condition (3.1).

Let $A = (a_{nk})$ be a $\mathcal{B}$-regular matrix, i.e. $A \in (c,\mathcal{B})_{reg}$. Then for each $x \in [0,2\pi]$, for which $f(x) \in BV[0,2\pi]$,

(3.4) $$\lim_n \sum_l \sum_k b_{nk}(i) a_{kl} \widetilde{s}_l(x) = \pi^{-1} g(x),$$

uniformly in i, if and only if

(3.5) $$\lim_n \sum_l \sum_k b_{nk}(i) a_{kl} \cos kt = 0,$$

for all $t \in [\delta,\pi]$, $\delta > 0$, uniformly in i.

4. LEMMAS

We need the following lemmas for the proof of our theorems.

Lemma 4.1. (Stieglitz [7], Satz 3).

Let $\mathcal{B} = (B_i)$ be a family of matrices with

(4.1) $$N(B_i) < \infty, \quad \text{for each } i.$$

Then A is $(c,\mathcal{B})_{reg}$ matrix (or $A \in (c,\mathcal{B})_{reg}$), if the following conditions are satisfied

(4.2) $$N(A) < \infty,$$

(4.3) there exists a whole number $r \geq 0$ such that

$$\sup_{0 \leq i < \infty,\, r \leq n \leq \infty} \sum_l \Big| \sum_k b_{nk}(i) a_{kl} \Big| < \infty,$$

(4.4) $$\lim_n \sum_l b_{nl}(i) a_{lk} = 0, \text{ uniformly in } i, k \text{ fixed}$$

(4.5) $$\lim_n \sum_l \sum_k b_{nl}(i) a_{lk} = 1, \text{ uniformly in } i$$

Lemma 4.2. (Jordan's convergence criterion for Fourier series, see [2])

If $f(x) \in BV(a,b)$ ((a,b) being some interval), then its Fourier series converges at every point of this interval. Its sum is $f(x)$ at a point of continuity and $[f(x+0) - f(x-0)]/2$ at a point of discontinuity of the first kind, Finally, if (a', b') lies entirely inside the interval (a,b), where $f(x)$ is continuous, then the Fourier series converges uniformly in (a', b').

5. PROOF OF THEOREM 3.1.

The partial sum $s_l'(x)$ of the series (1.3) is given by

$$s_l'(x) = \frac{1}{\pi} \int_0^\pi \psi_x(t) \left(\sum_{k=1}^{l} k \sin kt\right) dt$$

$$= -\frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{d}{dt} \left[\frac{\sin(l+\frac{1}{2})t}{2 \sin \frac{1}{2}t}\right] dt$$

$$= I_l + \frac{2}{\pi} \int_0^\pi \sin(l + \tfrac{1}{2})t \; d\, h_x(t),$$

where

$$I_l = \frac{1}{\pi} \int_0^\pi h_x(t) \frac{\sin(l+\frac{1}{2})}{\tan\frac{1}{2}t} \, dt.$$

Since $h_x(t) \in BV[0, \pi]$ and $h_x(t) \to h_x(0+)$, as $t \to 0, h_x(t) \cos \frac{1}{2}t$ has the same properties. Therefore, by Lemma 4.2

$$I_l \to h_x(0+), \text{ as } l \to \infty.$$

Now,

$$\sum_{l=0}^{\infty} \sum_{k=1}^{\infty} b_{nk}(i) a_{kl} s_l'(x)$$

(5.1) $$= \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} b_{nk}(i) a_{kl} I_l$$

$$+ \frac{2}{\pi} \int_0^\pi \left[\sum_{l=0}^{\infty} \sum_{k=1}^{\infty} b_{nk}(i) a_{kl} \sin(l + \frac{1}{2})t\right] d\, h_x(t)$$

$$= J_1 + J_2, \text{ say}.$$

Since A is $\mathcal{B}$-regular, by virtue of condition (4.5),

$$J_1 \to h_x(0+), \text{ as } n \to \infty, \text{ uniformly in } i.$$

Thus, we have to show that (3.2) holds if and only if

$$J_2 \to 0, \text{ as } n \to \infty, \text{ uniformly in } i.$$

Now, A being $\mathcal{B}$-regular, i.e. for all $x \in c, Ax \in c^{\mathcal{B}}$. Hence, there exists a whole number $r > 0$, and a constant M such that

$$\sup_{0 \le i < \infty, r \le n \le \infty} \left[\sum_{l=0}^{\infty} \left| \sum_{k=1}^{\infty} b_{nk}(i) a_{kl} \right| \right] < M. \tag{5.2}$$

Therefore,

$$\begin{aligned} &\left| \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} b_{nk}(i) a_{kl} \sin(l + \tfrac{1}{2})t \right| \\ &\le \sum_{l=0}^{\infty} \left| \sin(l + \tfrac{1}{2})t \right| \left| \sum_{k=1}^{\infty} b_{nk}(i) a_{kl} \right| \\ &\le \sum_{l=0}^{\infty} \left| \sum_{k=1}^{\infty} b_{nk}(i) a_{kl} \right| \\ &< M, \end{aligned} \tag{5.3}$$

for all $0 \le i < \infty$ and $r \le n < \infty$, for r as defined above.

Hence, by a theorem on the weak convergence in the Banach spaces of all continuous functions defined on a finite closed interval (see Banach [1], pp.134-135), it follows that (5.3) holds if and only if $J_2 \to 0$, as $n \to \infty$ and (3.3) holds.

Since (5.3) is satisfied, it follows that $J_2 \to 0$, if and only if (3.3) holds.

This completes the proof of Theorem 3.1.

6. PROOF OF THEOREM 3.2.

Partial sum of the series (1.2) is given by

$$\begin{aligned} \tilde{s}_l(x) &= \tfrac{1}{\pi} \int_0^{\infty} \psi_x(t) \sin lt \, dt \\ &= \tfrac{g(x)}{\pi} + \tfrac{1}{\pi} \int_0^{\pi} \cos lt \, d\, \psi_x(t). \end{aligned}$$

Therefore

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{nk}(i) a_{kl} \, \tilde{s}_l(x) = I_1 + I_2, \text{ say}$$

where

$$I_1 = \tfrac{1}{\pi}\sum_{k=0}^{\infty}\ \sum_{l=0}^{\infty} b_{nk}(i)g(x),$$

and

$$I_2 = \tfrac{1}{\pi}\int_0^{\pi}\left[\sum_{k=0}^{\infty} b_{nk}(i)a_{kl}\ \cos\ lt\right] d\,\psi_x(t).$$

By condition (4.5) of Lemma 4.1.,

$$I_1 \to g(x)/\pi, \text{ as } n \to \infty, \text{ uniformly in } i.$$

Thus, we have to show that

$$I_2 = \frac{1}{\pi}\int_0^{\pi} d\ \psi_x(t)K_{ni}(t) \to 0, \tag{6.1}$$

uniformly in i, as $n \to \infty$,
where

$$K_{ni}(t) = \sum_{k=0}^{\infty}\ \sum_{l=0}^{\infty} b_{nk}(i)a_{kl}\ \cos\ lt. \tag{6.2}$$

Now, it is easy to show that the condition (6.1) is equivalent to the following condition:

$$\tfrac{1}{\pi}\int_{\delta}^{\pi} K_{ni}(t)\ d\ \psi_x(t) \to 0, \tag{6.3}$$

uniformly in i, as $n \to \infty$, $0 < \delta < \pi$, for $f \in BV[0,2\pi]$ and for every $x \in [0,2\pi]$.

Therefore, arguing as in Theorem 3.1, we have

$$\begin{aligned}
|\ K_{ni}(t)\ | &= |\ \sum_{k=0}^{\infty}\ \sum_{l=0}^{\infty} b_{nk}(i)a_{kl}\ \cos\ lt| \\
&\le |\ \sum_{l=0}^{\infty}\ \cos\ lt \sum_{k=0}^{\infty} b_{nk}(i)a_{kl}| \\
&\le \sum_{l=0}^{\infty}\ |\ \cos\ lt\ \|\ \sum_{k=0}^{\infty} b_{nk}(i)a_{kl}| \\
&\le \sum_{l=0}^{\infty}\ |\ \sum_{k=0}^{\infty} b_{nk}(i)a_{kl}| \\
&< M,
\end{aligned} \tag{6.4}$$

for all $0 \le i < \infty$ and $r \le n < \infty$.

Hence, again by a theorem of weak convergence of sequence of Banach spaces of all continuous functions defined on a finite closed interval, it follows that (6.3) holds if and only if

(*i*) $|\ K_{ni}(t)\ | \le M$,

for all n and p, and $0 < \delta \le t \le \pi$; and (*ii*) (3.4) holds.

Since, by virtue of condition (3.4), (i) is true, it follows that (6.3) holds if and only if (3.4) holds. And (6.3) is equivalent to (6.1), hence (3.3) holds if and only if (6.1) holds.

This completes the proof of Theorem 3.2.

7. COROLLARIES

In the special cases in which $b_{nk}(i)$ are given by (1.9) and (1.10), we get the corresponding results for almost convergence and σ-convergence as the corollary given by Mursaleen ([5], Chapter VI, p.62) and Mursaleen ([5], Chapter VI, Theorem 2; also Theorem 2.3 of the present paper).

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