Commun. Fac. Sci. Univ. Ank. Series A1
V. 49. pp. 111 - 116 (2000)

SPECTRAL PROPERTIES OF NON-SELFADJOINT SYSTEM OF DIFFERENTIAL EQUATIONS

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(Received March 7, 2000; Accepted Sep. 6, 2000)

ABSTRACT

In this paper we investigate discrete spectrum of the boundary value problem

$$iy_1' + q_1(x)y_2 = \lambda y_1$$
$$-iy_2' + q_2(x)y_1 = \lambda y_2, \quad x \in R_+ = [0,\infty)$$
$$y_2(0) - hy_1(0) = 0$$

in the space $L_2(R_+, C^2)$, where q_i, i=1,2 are complex valued functions and $h \in C$.

1. INTRODUCTION

Let L denote the operator generated in $L_2(R_+)$ by the differential expression

$$l(y) = -y'' + q(x)y \ , \ x \in R_+ = [0,\infty)$$

and the boundary condition y(0)=0, where q is a complex valued function. The study of the spectral analysis of L was investigated by Naimark [9]. He proved that the spectrum of L consisted of the eigenvalues, the continuous spectrum and the spectral singularities. Pavlov[10] studied the dependence of the structure of the spectral singularities of L on the behaviour of q at infinity. The effect of the spectral singularities in the spectral expansion of L in terms of the principal functions have been investigated by Lyance[8]. The spectral singularities and the eigenfunction expansions of the quadratic pencil of the Schrödinger, Klein-Gordon and Dirac operators have been considered in [2]-[5].

Let us consider the boundary value problem (BVP)

$$iy_1' + q_1(x)y_2 = \lambda y_1$$
$$-iy_2' + q_2(x)y_1 = \lambda y_2, \quad x \in R_+ = [0,\infty) \tag{1.1}$$
$$y_2(0) - hy_1(0) = 0 \tag{1.2}$$

in the space $L_2(R_+, C^2)$, where the functions q_i, i=1,2 are complex valued continuous functions in R_+ and $h \in C$.

In this paper using the technique of the paper[3], we investigate the eigenvalues and the spectral singularities of the BVP (1.1)-(1.2), and prove that this BVP has a finite number of eigenvalues and spectral singularities and each of them is of a finite multiplicities.

Note that the properties of the principal functions corresponding to the eigenvalues and the spectral singularities of the BVP (1.1)-(1.2) have been investigated in [6].

2. Special Solutions of (1.1):

Let us suppose that

$$|q_i(x)| \leq c(1+x)^{-(1+\varepsilon)}, \ i=1,2, x \in R_+, \varepsilon > 0 \tag{2.1}$$

where c>0 is a constant.

We will denote the solutions of (1.1) satisfying the boundary conditions

$$y(x,\lambda) = \begin{pmatrix} e^{-i\lambda x} \\ 0 \end{pmatrix} [1+o(1)] \quad \lambda \in \overline{C}_- = \{\lambda : \lambda \in C, \operatorname{Im}\lambda \leq 0\}, x \to \infty$$

and

$$y(x,\lambda) = \begin{pmatrix} 0 \\ e^{i\lambda x} \end{pmatrix} [1+o(1)] \quad \lambda \in \overline{C}_+ = \{\lambda : \lambda \in C, \operatorname{Im}\lambda \geq 0\}, x \to \infty$$

by $E^-(x,\lambda)$ and $E^+(x,\lambda)$, respectively

Under the condition (2.1) the solutions $E^-(x,\lambda)$ and $E^+(x,\lambda)$ of (1.1) exist, are unique and have the representations

$$E^-(x,\lambda) := \begin{pmatrix} e_1^-(x,\lambda) \\ e_2^-(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{-i\lambda x} + \int_x^\infty H_{11}(x,t) e^{-i\lambda t} dt \\ \int_x^\infty H_{21}(x,t) e^{-i\lambda t} dt \end{pmatrix}, \lambda \in \overline{C}_- \tag{2.2}$$

$$E^+(x,\lambda) := \begin{pmatrix} e_1^+(x,\lambda) \\ e_2^+(x,\lambda) \end{pmatrix} = \begin{pmatrix} \int_x^\infty H_{12}(x,t) e^{i\lambda t} dt \\ e^{i\lambda x} + \int_x^\infty H_{22}(x,t) e^{i\lambda t} dt \end{pmatrix}, \lambda \in \overline{C}_+, \tag{2.3}$$

where the functions $H_{ij}(x,t)$, i,j=1,2 are solutions of the system of Volterra integral equations and

$$\left|H_{ij}(x,t)\right| \le c\sum_{k=1}^{2}\left|q_k\left(\frac{x+t}{2}\right)\right|, \quad i,j=1,2 \tag{2.4}$$

where $c>0$ is a constant [1]. Moreover the functions $E^{-}(x,\lambda)$ and $E^{+}(x,\lambda)$ are analytic with respect to λ in $C_{+}=\{\lambda:\lambda\in C, \operatorname{Im}\lambda>0\}$ and $C_{-}=\{\lambda:\lambda\in C, \operatorname{Im}\lambda<0\}$, respectively and continuous up to the real axis.

3. Discrete Spectrum of BVP (1.1)-(1.2):

Let us consider the functions

$$a^{+}(\lambda)=e_2^{+}(0,\lambda)-he_1^{+}(0,\lambda)$$

$$a^{-}(\lambda)=e_2^{-}(0,\lambda)-he_1^{-}(0,\lambda)$$

We will denote the set of all eigenvalues and spectral singularities of the BVP (1.1)-(1.2) by σ_d and σ_{ss}, respectively.

We can easily prove that

$$\sigma_d=\{\lambda:\lambda\in C_{+}, a^{+}(\lambda)=0\}\cup\{\lambda:\lambda\in C_{-}, a^{-}(\lambda)=0\} \tag{3.1}$$

$$\sigma_{ss}=\{\lambda:\lambda\in R^{*}, a^{+}(\lambda)=0\}\cup\{\lambda:\lambda\in R^{*}, a^{-}(\lambda)=0\} \tag{3.2}$$

where $R^{*}=R\backslash\{0\}$ [2].

From (3.1) and (3.2) we see that in order to investigate the structure of the discrete spectrum of the BVP (1.1)-(1.2) we need to discuss the structure of zeros of a^{+} and a^{-} in $\overline{C}_{+}$ and $\overline{C}_{-}$, respectively. For the sake of simplicity we will consider only the zeros of a^{+} in $\overline{C}_{+}$

Let us define

$$P_1^{\mp}=\{\lambda:\lambda\in C_{\mp}, a^{+}(\lambda)=0\} \quad P_2^{\mp}=\{\lambda:\lambda\in R, a^{+}(\lambda)=0\}$$

It follows from (3.1) and (3.2) that

$$\sigma_d=P_1^{+}\cup P_1^{-}, \quad \sigma_{ss}=\{P_1^{+}\cup P_1^{-}\}\backslash\{0\} \tag{3.3}$$

Lemma 3.1. If (2.1) holds, then

(i) The set P_1^{+} is bounded and has at most a countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis.

(ii) The set P_2^{+} is compact and its linear Lebesque measure is zero.

Proof. (2.3) yield that a^{+} is analytic in C_{+}, continuous in $\overline{C}_{+}$ and has the form

$$a^{+}(\lambda)=1+\int_0^{\infty}[H_{22}(0,t)-hH_{12}(0,t)]e^{i\lambda t}dt \tag{3.4}$$

Hence (3.4) implies that

$$a^{+}(\lambda)=1+o(1), \quad \lambda\in\overline{C}_{+}, \quad |\lambda|\to\infty \tag{3.5}$$

which shows the boundedness of the sets P_1^+ and P_2^+. The proof of lemma is a direct consequence of (3.5) and uniqueness of analytic functions ([7]).

From Lemma 3.1 we get the following.

Theorem 3.2. Under the condition (2.1) we have

(i) The set of eigenvalues of the BVP (1.1)-(1.2) is bounded, is no more than countable and its limit point can lie only in a bounded subinterval of the real axis.

(ii) The set of spectral singularities of the BVP (1.1)-(1.2) is bounded and its linear Lebesgue measure is zero.

Definition 3.3. The multiplicity of zero a^+ (or a^-) in $\overline{C}_+$ (or $\overline{C}_-$) is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.1)-(1.2).

Theorem 3.4. If

$$|q_i(x)| \le ce^{-\varepsilon x}, \; i=1,2, \; x\in R_+, \; \varepsilon>0 \tag{3.6}$$

holds; then the BVP (1.1)-(1.2) has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

Proof. From (2.4) we find that

$$\left|H_{ij}(0,t)\right| \le ce^{-\frac{\varepsilon}{2}t}. \tag{3.7}$$

(3.4) and (3.7) shows that, the functions a^+ has an analytic continuation from the real axis to the half plane $\operatorname{Im}\lambda > -\frac{\varepsilon}{2}$. So the limit points of the sets P_1^+ and P_2^+ can not lie in R, i.e., the bounded sets P_1^+ and P_2^+ have no limit points. Therefore, we have the finiteness of the zeros of a^+ in $\overline{C}_+$. Moreover all zeros of a^+ in $\overline{C}_+$ has a finite multiplicity. Similarly we get that the function a^- has a finite number of zeros with finite multiplicity in $\overline{C}_-$.

It is seen that the condition (3.6) guaranties of the analytic continuation of a^+ and a^- from the real axis to lower and upper half-planes, respectively. So the finiteness of eigenvalues and spectral singularities of the BVP (1.1)-(1.2) are obtained as a result of this analytic continuations.

Now let us suppose that

$$|q_i(x)| \le ce^{-\varepsilon x^\alpha}, \; i=1,2, \; x\in R_+, \; \varepsilon>0, \; \frac{1}{2}\le\alpha<1 \tag{3.8}$$

hold, which is weaker than (3.6). It is evident under the condition (3.8) that the function a^+ does not have an analytic continuation from the real axis to lower half-plane. Similarly a^- does not have an analytic continuation from the real axis to upper half-plane. Therefore under the condition (3.8) the finiteness of eigenvalues and spectral singularities of the BVP (1.1)-(1.2) cannot be proved by the same technique used in Theorem 3.4. Let us denote the set of all limit points of P_1^+ and P_2^+ by

P_3^+ and P_4^+, respectively, and the set of all zeros of a^+ with *infinite multiplicity* in $\overline{C}_+$ by P_5^+.

It is clear that

$$P_1^+ \cap P_5^+ = \varnothing \ , \ P_3^+ \subset P_2^+, P_4^+ \subset P_2^+, P_5^+ \subset P_2^+$$

and the linear Lebesgue measures of P_3^+, P_4^+ and P_5^+ are zero. Using the continuity of all derivatives of a^+ on the real axis we obtain

$$P_3^+ \subset P_5^+ \ , \ P_4^+ \subset P_5^+ . \tag{3.9}$$

Lemma 3.5. If (3.8) holds, then $P_5^+ = \varnothing$.

Proof. There exist a T>0 such that

$$\left|\frac{d^n}{d\lambda^n} a^+(\lambda)\right| \le c_n^+ \ , n=0,1,... \ , \lambda \in \overline{C}_+, |\lambda| < T$$

hold, where, c_n^+, n=0,1,... are constants. By Pavlov's theorem, we get

$$\int_0^h \ln F(s) d\mu\left(P_{5,s}^+\right) > -\infty \tag{3.10}$$

where $F(s) = \inf_n \frac{c_n^+ s^n}{n!}$, $\mu(P_{5,s}^+)$ is the linear Lebesque measure of s-neighbourhood of P_5^+ and h>0 is a constant ([3],[10]).

Using (2.4) and (3.8) we obtain

$$c_n^+ = 2^n c \int_0^\infty x^n e^{-\varepsilon x^\alpha} dx \le B b^n n! n^{n\frac{1-\alpha}{\alpha}} \tag{3.11}$$

where B and b are constants depending ε, α and c. Substituting (3.11) in the definition of F(s) we arrive at

$$F(s) = \inf_n \frac{c_n^+ s^n}{n!} \le B \exp\left\{ -\frac{1-\alpha}{\alpha} e^{-\frac{1}{1-\alpha}} b^{-\frac{\alpha}{1-\alpha}} s^{-\frac{\alpha}{1-\alpha}} \right\}$$

or

$$\int_0^h s^{-\frac{\alpha}{1-\alpha}} d\mu(P_{5,s}^+) < \infty \tag{3.12}$$

by (310). So $\frac{\alpha}{1-\alpha} \ge 1$ hence (3.12) holds for arbitrary s if and only if $\mu(P_{5,s}^+) = 0$ or $P_5^+ = \varnothing$.

Theorem 3.6: Under the condition (3.8) the BVP (1.1)-(1.2) has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.

Proof. To be able to prove the theorem we have to show that the functions a^+ and a^- has a finite number of zeros with finite multiplicities in $\overline{C}_+$ and $\overline{C}_-$, respectively. We will prove it only for a^+.

From (3.9) and Lemma 3.5 we find that $P_3^+ = P_4^+ = \varnothing$. So the bounded sets P_1^+ and P_2^+ have no limit points, i.e., the function a^+ has only a finite number of zeros in $\overline{C}_+$. Since $P_5^+ = \varnothing$ these zeros are of finite multiplicity.

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