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ON A CLASS OF SURFACES IN THE EUCLIDEAN SPACE

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ABSTRACT

In this study we consider surfaces in the Euclidean space $\mathbf{E}^{2+d}$ satisfying the conditions $\overline{R}\cdot h = L_h Q(g,h)$ and $\overline{R}\cdot h = L_{A_H}\hat{Q}(A_H,h)$.

1. INTRODUCTION

Let M be an n-dimensional submanifold in (n + d)-dimensional Euclidean space $\mathbf{E}^{n+d}$. We denote the Euclidean metric on $\mathbf{E}^{n+d}$ by g and the Levi-Civita connection of g by $\widetilde{\nabla}$. The induced metric on M is also denoted by g and the Levi-Civita connection of (M, g) by ∇. Then the *second fundamental form* h of M in $\mathbf{E}^{n+d}$ is defined by the formula of Gauss: $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$, where X, Y are vector fields tangent to M and h(X,Y) is the second fundamental form of M. Let ξ be an arbitrary normal vector field on M. Then the *shape operator* A_ξ of M with respect to ξ and the normal connection $\nabla^\perp$ of M in $\mathbf{E}^{n+d}$ are defined by the formula of Weingarten: $\widetilde{\nabla}_X\xi = -A_\xi X + \nabla^\perp_X\xi$, where by $-A_\xi X$ and $\nabla^\perp_X\xi$ are respectively the tangential and the normal components of $\widetilde{\nabla}_X\xi$. The second fundemental form h and the shape operator A_ξ are related by $g(A_\xi X, Y) = g(h(X, Y), \xi)$. If h = 0 then M is called *totally geodesic*. M is *totally umbilical* if all shape operators are proportional to the identity map. M is an isotropic immersion if for each p in M, $\|h(X,X)\|$ is independent of the choice of a unit vector X in T_pM. The *mean curvature vector* H of M is given by $H = \frac{1}{n}tr(h)$ and the *mean curvature* of M is defined by $\alpha := \sqrt{g(H,H)}$. Let $X \wedge Y$ denote the endomorphism $Z \rightarrow g(Y,Z)X - g(X,Z)Y$. Then the curvature tensor R of M is given by the equation of Gauss:

$$R(X,Y)Z = \sum_{i=1}^{d}(A_i X \wedge A_i Y)Z \,, \tag{1.1}$$

where $A_i := A_{\xi_i}$ and $\{\xi_1,\ldots,\xi_d\}$ is a local orthonormal basis for the normal space $T^{\perp}M$. If M is a surface then the Gaussian curvature K of M at $p \in M$ is $K(p) := g(R(X,Y)Y, X)$, where $\{X, Y\}$ form an orthonormal basis of the tangent space T_pM. The equation of Ricci becomes $g(R^{\perp}(X, Y)\xi, \eta) = g([A_\xi, A_\eta]X, Y)$ for ξ and η vectors normal to M, where $R^{\perp}$ is the curvature tensor of the normal connection $\nabla^{\perp}$. If $R^{\perp} = 0$ then M is said to have *trivial normal connection* [1].

2. SEMIPARALLEL SURFACES

Let $\overline{\nabla}$ be the connection of van der Waerden-Bortolotti of M, i. e., $\overline{\nabla}$ is the connection in $TM \oplus T^{\perp}M$ built with ∇ and $\widetilde{\nabla}$. Denote the curvature tensor of $\overline{\nabla}$ by $\overline{R}$.

Definition 2.1. Let M be an n-dimensional submanifold in (n + d)-dimensional Euclidean space $\mathbf{E}^{n+d}$. M is called *semiparallel* if

$$\overline{R}\ (X, Y)\cdot h = \overline{\nabla}_X \overline{\nabla}_Y h - \overline{\nabla}_Y \overline{\nabla}_X h - \overline{\nabla}_{[X,Y]} h = 0$$

that is,

$$(\overline{R}(X, Y)\cdot h)(U,V) = R^{\perp}(X,Y)h(U,V) - h(R(X, Y)U, V) - h(U, R(X, Y)V) = 0 \tag{2.1}$$

for all vector fields X,Y,U,V tangent to M [2] (see also [4]).

In the present section we consider that M is a surface in the Euclidean space $\mathbf{E}^{2+d}$.

Teorem 2.2. [2]. Let M be a surface in the Euclidean space $\mathbf{E}^{2+d}$. Then M is semiparallel if and only if locally

i) M is totally umbilical, i. e., M is an open part of a sphere S^2 in $\mathbf{E}^3 \subset \mathbf{E}^{2+d}$, or

ii) M is a flat surface with trivial normal connection, or

iii) M is an isotropic surface with codimension at least 3 and $\alpha^2 = 3K$.

Proposition 2.3. Let M be a surface in the Euclidean space $\mathbf{E}^{2+d}$. Then there is an orthonormal basis $\{e_1, e_2\}$ of T_pM such that

$$A_{\xi_1} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix},\quad A_{\xi_2} = \begin{bmatrix} a_2 & b_2 \\ b_2 & -a_2 \end{bmatrix},\quad A_{\xi_3} = \begin{bmatrix} a_3 & b_3 \\ b_3 & -a_3 \end{bmatrix},$$

where ξ_1, ξ_2 and ξ_3 are vector fields normal to M.

Lemma 2.4. [2]. Let M be a surface in the Euclidean space $\mathbf{E}^{2+d}$ and $\{e_1, e_2\}$ be an orthonormal basis of the tangent space T_pM. Then

$$(\overline{R}(e_1,e_2)\cdot h)(e_1, e_1) = [\lambda-\mu][a_2b_2 + a_3b_3]\xi_1 + [-\lambda(\lambda-\mu)b_2 + 2\beta a_3 + 2Kb_2]\,\xi_2 + [-\lambda(\lambda-\mu)b_3 - 2\beta a_2 + 2Kb_3]\,\xi_3, \tag{2.2}$$

$$(\overline{R}(e_1,e_2)\cdot h)(e_1, e_2) = [\lambda-\mu][\,b_2^2 + b_3^2 - K]\xi_1 + 2[\beta b_3 - Ka_2]\,\xi_2 - 2\,[\beta b_2 + Ka_3]\,\xi_3 \tag{2.3}$$

and

$$(\overline{R}(e_1,e_2)\cdot h)(e_2, e_2) = -[\lambda-\mu][a_2b_2 + a_3b_3]\xi_1 + [-\mu(\lambda-\mu)b_2 - 2\beta a_3 - 2Kb_2]\,\xi_2 + [-\mu(\lambda-\mu)b_3 + 2\beta a_2 - 2Kb_3]\,\xi_3, \quad (2.4)$$

where K is the Gaussian curvature of M in $\mathbf{E}^{2+d}$ and $\beta = a_2b_3 - a_3b_2$.

Definition 2.5. Let M be a submanifold in the Euclidean space $\mathbf{E}^{n+d}$. We define a tensor denoted by Q as

$$Q(g, h)(e_i, e_j; e_k, e_l) = = - h((e_k \wedge e_l)\, e_i, e_j) - h(e_i, (e_k \wedge e_l)\, e_j) \quad (2.5)$$

for vector fields e_i, e_j, e_k, e_l tangent to M (see [3]).

If the tensors $\overline{R}\cdot h$ and Q(g, h) are linearly dependent then M is called *extended semiparallel*, that is, the equality

$$\overline{R}\cdot h = L_h Q(g, h)$$

holding on the set

$$U_h = \{p\in M : Q(g, h) \neq 0\}$$

where L_h is some function on U_h.

Lemma 2.6. Let M be a surface in the Euclidean space $\mathbf{E}^{2+d}$ and $\{e_1, e_2\}$ be an orthonormal basis of the tangent space T_pM. Then

$$Q(g, h)(e_1, e_1; e_1, e_2) = 2(b_2\xi_2 + b_3\xi_3), \quad (2.6)$$

$$Q(g, h)(e_1, e_2; e_1, e_2) = -(\lambda-\mu)\xi_1 - 2a_2\xi_2 - 2a_3\xi_3 \quad (2.7)$$

and

$$Q(g, h)(e_2, e_2; e_1, e_2) = -2(b_2\xi_2 + b_3\xi_3). \quad (2.8)$$

Proof. Let $\{e_1, e_2\}$ be an orthonormal basis of the tangent space T_pM. Since the dimension of the first normal space $N_p^1(M)$:=span$\{h(X,Y) \mid X,Y\in T_pM\}$ is at most 3 we can choose normal vectors $\xi_1,\ldots,\xi_d$ such that $A_{\xi_4} = \ldots = A_{\xi_d} = 0$. Suppose ξ_1 is in the direction of the mean curvature vector H. So by the use of Proposition 2.3, we have

$$h(e_1, e_1) = \lambda\xi_1 + a_2\xi_2 + a_3\xi_3, \quad (2.9)$$

$$h(e_1, e_2) = b_2\xi_2 + b_3\xi_3 \quad (2.10)$$

and

$$h(e_2, e_2) = \mu\xi_1 - a_2\xi_2 - a_3\xi_3. \quad (2.11)$$

Therefore using (2.5) and the symmetrization of the second fundamental form h we get the result.

Lemma 2.7. Let M be a surface in the Euclidean space $\mathbf{E}^{2+d}$. If M is extended semiparallel then $R^{\perp}(e_1, e_2)H = 0$ for all vectors e_1, e_2 tangent to M.

Proof. Let ξ be a normal vector at a point $p\in M$. Choose an orthonormal basis $\{e_1, e_2\}$ of T_pM such that $A_\xi e_i = \lambda_i e_i$ $(1 \leq i \leq 2)$ i. e., e_1, e_2 are eigenvectors of A_ξ. Since M is extended semiparallel, we can write

$$(\overline{R}(e_1,e_2)\cdot h)(e_i, e_i) = L_h Q(g, h)(e_i, e_i; e_1,e_2).$$

So using (2.1) we have

$$R^{\perp}(e_1,e_2)h(e_i,e_i) = 2h(R(e_1,e_2)e_i, e_i) - 2L_h g(e_2, e_i)h(e_1,e_i) + 2\, L_h g(e_1, e_i)h(e_2,e_i). \quad (2.12)$$

Since

$$g(R^{\perp}(e_1,e_2)H, \xi) = \frac{1}{2}\sum_{i=1}^{2} g(R^{\perp}(e_1,e_2)h(e_i,e_i),\xi) \tag{2.13}$$

then substituting (2.12) into (2.13) we get

$$g(R^{\perp}(e_1,e_2)H, \xi) = \sum_{i=1}^{2}[g(h(R(e_1,e_2)e_i,e_i),\xi) \; . \\ - L_h g(e_2, e_i)g(h(e_1,e_i),\xi) + L_h g(e_1, e_i)g(h(e_2,e_i),\xi)]. \tag{2.14}$$

Since $g(A_\xi e_i, e_j) = g(h(e_i, e_j), \xi)$, the equation (2.14) can be written as

$$g(R^{\perp}(e_1,e_2)H, \xi) = g(R(e_1,e_2)e_1, A_\xi e_1) + g(R(e_1,e_2)e_2, A_\xi e_2) \\ - L_h g(e_2, e_1)g(e_1, A_\xi e_1) - L_h g(e_2, e_2)g(e_1, A_\xi e_2) \\ + L_h g(e_1, e_1) g(e_2, A_\xi e_1) + L_h g(e_1, e_2) g(e_2, A_\xi e_2). \tag{2.15}$$

So substituting $A_\xi e_i = \lambda_i e_i$ into (2.15) we obtain $g(R^{\perp}(e_1,e_2)H, \xi) = 0$. Since ξ is an arbitrary normal vector the last equality gives us $R^{\perp}(e_1,e_2)H = 0$. This completes the proof of the Lemma.□

Theorem 2.8. Let M be a surface in the Euclidean space $\mathbf{E}^{2+d}$. If M is extended semiparallel then locally

i) M is semiparalel, or

ii) M is a surface in $\mathbf{E}^{2+d}$ with trivial normal connection which has the Gaussian curvature $K = L_h$, or

iii) M is an isotropic surface with codimension at least 3 and $\alpha^2 = 3K - 2L_h$.

Proof. If M is 2-semiparallel then $\overline{R}\cdot h = 0$. Thus the condition $\overline{R}\cdot h = L_h Q(g, h)$ is trivially satisfied. So we consider the case M is not 2-semiparallel.

Suppose $\overline{R}\cdot h \neq 0$ and $\overline{R}\cdot h = L_h Q(g, h)$ are satisfied on M. So by the use of (2.2)-(2.4) and (2.6)-(2.8), the conditions

$$(\overline{R}(e_1,e_2)\cdot h)(e_1, e_1) = L_h Q(g, h)(e_1, e_1; e_1,e_2),$$
$$(\overline{R}(e_1,e_2)\cdot h)(e_2, e_2) = L_h Q(g, h)(e_2, e_2; e_1,e_2)$$

and

$$(\overline{R}(e_1,e_2)\cdot h)(e_1, e_2) = L_h Q(g, h)(e_1, e_2; e_1,e_2)$$

give that

$$[\lambda-\mu][a_2b_2 + a_3b_3] = 0, \tag{2.16}$$
$$-\lambda(\lambda-\mu)b_2 + 2\beta a_3 + 2b_2(K - L_h) = 0, \tag{2.17}$$
$$-\lambda(\lambda-\mu)b_3 - 2\beta a_2 + 2b_3(K - L_h) = 0, \tag{2.18}$$
$$[\lambda-\mu][b_2^2 + b_3^2 - K + L_h] = 0, \tag{2.19}$$
$$\beta b_3 - (K - L_h)a_2 = 0, \tag{2.20}$$
$$\beta b_2 + (K - L_h)a_3 = 0, \tag{2.21}$$
$$-\mu(\lambda-\mu)b_2 - 2\beta a_3 - 2b_2(K - L_h) = 0 \tag{2.22}$$

and

$$-\mu(\lambda-\mu)b_3 + 2\beta a_2 - 2b_3(K - L_h) = 0. \tag{2.23}$$

Since ξ_1 is in the direction of the mean curvature vector H, by Lemma 2.7, we obtain $R^{\perp}(e_1,e_2)\xi_1 = 0$ which gives

$$(\lambda-\mu)b_2 = 0 \text{ and } (\lambda-\mu)b_3 = 0. \tag{2.24}$$

Firstly we consider the case $\lambda \neq \mu$. So we have $b_2 = b_3 = 0$. By Proposition 2.3, since the shape operators of M are diagonazible then $R^{\perp} = 0$, i.e., M has trivial normal connection and so the equation (2.19) gives the Gaussian curvature of M is $K = L_h$.

Now suppose $\lambda = \mu$. We can choose a basis of the tangent space T_pM denoted by $\{\tilde{e}_1, \tilde{e}_2\}$ such that $A_{\xi_1}\tilde{e}_1 = \lambda_1 \tilde{e}_1$ and $A_{\xi_2}\tilde{e}_2 = \lambda_2 \tilde{e}_2$. That is, $\tilde{e}_1$ and $\tilde{e}_2$ are eigenvectors of A_{ξ_1} and A_{ξ_2}. So by Proposition 2.3, $b_2 = 0$ and $\beta = \tilde{a}_2 b_3$. Therefore we can rewrite the equations (2.16)-(2.23) as

$$\tilde{a}_2 b_3 a_3 = 0, \tag{2.25}$$

$$[-\tilde{a}_2^2 + (K - L_h)]b_3 = 0, \tag{2.26}$$

$$[b_3^2 - (K - L_h)]\tilde{a}_2 = 0, \tag{2.27}$$

$$(K - L_h)a_3 = 0, \tag{2.28}$$

So we have the following cases:

a) Suppose $b_3 = \tilde{a}_2 = a_3 = 0$. Then by Proposition 2. 3, we get

$$A_{\xi_1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad A_{\xi_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{\xi_3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which gives M is totaly umbilical. But in this case $\overline{R} \cdot h = 0$. This contradicts our assumption $\overline{R} \cdot h \neq 0$. So this case cannot occur.

b) Now suppose $b_3 = 0$, $\tilde{a}_2 \neq 0$. Then by Proposition 2.3, we get

$$A_{\xi_1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad A_{\xi_2} = \begin{bmatrix} \tilde{a}_2 & 0 \\ 0 & -\tilde{a}_2 \end{bmatrix}, \quad A_{\xi_3} = \begin{bmatrix} \tilde{a}_2 & 0 \\ 0 & \tilde{a}_2 \end{bmatrix}.$$

Since the shape operators of M are diagonazible then $R^{\perp} = 0$ and using (2.27) we obtain $K = L_h$.

c) Now we consider $b_3 \neq 0$. If $\tilde{a}_2 = 0$ then by (2.26) we obtain $K = L_h$. If $\tilde{a}_2 \neq 0$ then by (2.25), $a_3 = 0$. So by Proposition 2.3, we have

$$A_{\xi_1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad A_{\xi_2} = \begin{bmatrix} \tilde{a}_2 & 0 \\ 0 & -\tilde{a}_2 \end{bmatrix}, \quad A_{\xi_3} = \begin{bmatrix} 0 & \tilde{a}_2 \\ \tilde{a}_2 & 0 \end{bmatrix},$$

which gives $K = \lambda^2 - 2\tilde{a}_2^2$. By the use of (2.26) we obtain $3K - 2L_h = \lambda^2$. In this case $\dim N_p^1(M) = 3$ and $\|h(X, X)\|$ is independent of the choice of a unit vector X in T_pM. It is easy to see that $\alpha^2 = 3K - 2L_h$ at p.Hence we get the result, as required.

3. SURFACES SATISFYING THE CONDITION $\overline{R}\cdot h = L_{A_H}\hat{Q}(A_H,h)$

Definition 3.1. Let M be a submanifold in the Euclidean space $\mathbf{E}^{n+d}$. We define a tensor denoted by $\hat{Q}$ as

$$\hat{Q}(A_H, h)(e_i, e_j; e_k, e_l) = -h((e_k \wedge_{A_H} e_l)e_i, e_j) - h(e_i, (e_k \wedge_{A_H} e_l)e_j \tag{3.1}$$

where

$$(e_k \wedge_{A_H} e_l)e_i = g(A_H e_1, e_i)e_k - g(A_H e_k, e_i)e_1, \tag{3.2}$$

(see [5] and [6]).

In the present section we consider surfaces in the Euclidean space $\mathbf{E}^{2+d}$ satisfying the condition

$$\overline{R}\cdot h = L_{A_H}\hat{Q}(A_H,h) \tag{3.3}$$

holding on the set

$$U_{A_H} = \{p\in M : \hat{Q}(A_H,h) \neq 0\}$$

where L_{A_H} is some function on $U_{A_H} \subset M$.

Lemma 3.2. Let M be a surface in the Euclidean space $\mathbf{E}^{2+d}$ and $\{e_1, e_2\}$ be an orthonormal basis of the tangent space T_pM. If M satisfies the condition $\overline{R}\cdot h = L_{A_H}\hat{Q}(A_H,h)$ then

$$\hat{Q}(A_H, h)(e_1, e_1; e_1, e_2) = 2\lambda(\lambda+\mu)(b_2\xi_2 + b_3\xi_3), \tag{3.4}$$

$$\hat{Q}(A_H, h)(e_1, e_2; e_1, e_2) = -(\lambda+\mu)^2(a_2\xi_2 + a_3\xi_3) \tag{3.5}$$

and

$$\hat{Q}(A_H, h)(e_2, e_2; e_1, e_2) = -2\mu(\lambda+\mu)(b_2\xi_2 + b_3\xi_3). \tag{3.6}$$

Proof. Using (3.1) and the symmetrization of the second fundamental form h we get (3.4)-(3.6).

Lemma 3.3. Let M be a surface in the Euclidean space $\mathbf{E}^{2+d}$. If M satisfies the condition $\overline{R}\cdot h = L_{A_H}\hat{Q}(A_H,h)$ then $R^{\perp}(e_1, e_2)H = 0$ for all vectors e_1, e_2 tangent to M.

Proof. Similar to the proof of Lemma 2.7.

Theorem 3.4. Let M be a surface in the Euclidean space $\mathbf{E}^{2+d}$. If M satisfies the condition $\overline{R}\cdot h = L_{A_H}\hat{Q}(A_H,h)$ then locally

i) M is semiparalel, or

ii) M is a surface in $\mathbf{E}^{2+d}$ with trivial normal connection which has the Gaussian curvature $K = 2\alpha^2 L_{A_H}$, or

iii) M is an isotropic surface with codimension at least 3 in $\mathbf{E}^{2+d}$ satisfying $3K = (1 + 4L_{A_H})\alpha^2$.

Proof. If M is 2-semiparallel then $\overline{R}\cdot h = 0$. Thus the condition $\overline{R}\cdot h = L_{A_H}\hat{Q}(A_H, h)$ is trivially satisfied. So we consider the case M is not 2-semiparallel.

Suppose $\overline{R}\cdot h \neq 0$ and $\overline{R}\cdot h = L_{A_H}\hat{Q}(A_H, h)$ are satisfied on M. So by the use of (2.2)-(2.4) and (3.4)-(3.6), the conditions

$$(\overline{R}(e_1,e_2)\cdot h)(e_1, e_1) = L_{A_H}\hat{Q}(A_H, h)(e_1, e_1, e_1, e_2),$$

$$(\overline{R}(e_1,e_2)\cdot h)(e_2, e_2) = L_{A_H}\hat{Q}(A_H, h)(e_2, e_2, e_1, e_2),$$

and

$$(\overline{R}(e_1,e_2)\cdot h)(e_1, e_2) = L_{A_H}\hat{Q}(A_H, h)(e_1, e_2, e_1, e_2),$$

give that

$$[\lambda-\mu][a_2b_2 + a_3b_3] = 0, \quad (3.7)$$
$$-\lambda(\lambda-\mu)b_2 + 2\beta a_3 + 2b_2[K - L_{A_H}\lambda(\lambda+\mu)] = 0, \quad (3.8)$$
$$-\lambda(\lambda-\mu)b_3 - 2\beta a_2 + 2b_3[K - L_{A_H}\lambda(\lambda+\mu)] = 0, \quad (3.9)$$
$$[\lambda-\mu][b_2^2 + b_3^2 - K] = 0, \quad (3.10)$$
$$2\beta b_3 - (2K - L_{A_H}(\lambda+\mu)^2)a_2 = 0, \quad (3.11)$$
$$-2\beta b_2 - (2K - (\lambda+\mu)^2)a_3 = 0, \quad (3.12)$$
$$-\mu(\lambda-\mu)b_2 - 2\beta a_3 - 2b_2(K - L_{A_H}\mu(\lambda+\mu)) = 0 \quad (3.13)$$

and

$$-\mu(\lambda-\mu)b_3 + 2\beta a_2 - 2b_3(K - L_{A_H}\mu(\lambda+\mu)) = 0. \quad (3.14)$$

Since ξ_1 is in the direction of the mean curvature vector H, by Lemma 3.3, we obtain $R^{\perp}(e_1,e_2)\xi_1 = 0$ which gives

$$(\lambda-\mu)b_2 = 0 \text{ and } (\lambda-\mu)b_3 = 0. \quad (3.15)$$

Firstly we consider the case $\lambda \neq \mu$. So we have $b_2 = b_3 = 0$. By Proposition 2.3, since the shape operators of M are diagonazible then $R^{\perp} = 0$, i.e., M has trivial normal connection and the equation (3.10) gives the Gaussian curvature of M is K = 0, which gives M is semiparallel. But this contradicts our assumption M is not semiparallel. So this case can not occur.

Now suppose $\lambda = \mu$. Similar to the proof of Theorem 2.8 we can choose a basis of the tangent space T_pM denoted by $\{\widetilde{e}_1, \widetilde{e}_2\}$ such that $A_{\xi_1}\widetilde{e}_1 = \lambda_1\widetilde{e}_1$ ve $A_{\xi_2}\widetilde{e}_2 = \lambda_2\widetilde{e}_2$. That is, $\widetilde{e}_1$ and $\widetilde{e}_2$ are eigenvectors of A_{ξ_1} and A_{ξ_2}. So by Proposition 2. 3, $b_2 = 0$ and $\beta = \widetilde{a}_2 b_3$. Therefore we can rewrite the equations (3.7)-(3.14) as

$$\widetilde{a}_2 b_3 a_3 = 0, \quad (3.16)$$
$$[-\widetilde{a}_2^2 + K - 2L_{A_H}\lambda^2]b_3 = 0, \quad (3.17)$$
$$[b_3^2 - K + 2L_{A_H}\lambda^2]a_2 = 0, \quad (3.18)$$
$$[K - 2L_{A_H}\lambda^2]a_3 = 0, \quad (3.19)$$

So we have the following cases:

a) Suppose $b_3 = \widetilde{a}_2 = a_3 = 0$. Then by Proposition 2.3, we get

$$A_{\xi_1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad A_{\xi_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{\xi_3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which gives M is totaly umbilical. But in this case $\overline{R} \cdot h = 0$. This contradicts our assumption $\overline{R} \cdot h \neq 0$.

b) Suppose $b_3 = 0$, $\widetilde{a}_2 \neq 0$. Then by Proposition 2.3, we get

$$A_{\xi_1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad A_{\xi_2} = \begin{bmatrix} \widetilde{a}_2 & 0 \\ 0 & -\widetilde{a}_2 \end{bmatrix}, \quad A_{\xi_3} = \begin{bmatrix} \widetilde{a}_2 & 0 \\ 0 & \widetilde{a}_2 \end{bmatrix}.$$

Since the shape operators of M are diagonazible then $R^{\perp} = 0$ and using (3.18) we obtain $K = 2 L_{A_H} \lambda^2$ at p. Since $\lambda^2 = \alpha^2$, by the use of (3.17) we obtain $K = 2\alpha^2 L_{A_H}$.

c) Now we consider $b_3 \neq 0$. If $\widetilde{a}_2 = 0$ then using (3.17) we obtain $K = 2 L_{A_H} \lambda^2$, which gives $K = 2\alpha^2 L_{A_H}$.

If $\widetilde{a}_2 \neq 0$ then by (3.16), $a_3 = 0$. So by Proposition 2.3, we have

$$A_{\xi_1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad A_{\xi_2} = \begin{bmatrix} \widetilde{a}_2 & 0 \\ 0 & -\widetilde{a}_2 \end{bmatrix}, \quad A_{\xi_3} = \begin{bmatrix} 0 & b_3 \\ b_3 & 0 \end{bmatrix},$$

which gives $K = \lambda^2 - \widetilde{a}_2^2 - b_3^2$. In this case $\dim N_p^1(M) = 3$ and $\|h(X,X)\|$ is independent of the choice of a unit vector X in T_pM. So using (3.17) and (3.18) it is easy to see that $3K = (1 + 4 L_{A_H})\alpha^2$ at p. This completes the proof of the theorem.

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