

ON THE MODULAR INTEGRALS AND THEIR MELLIN TRANSFORMS

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(Received: Oct. 17, 2002 ; Revised: March 21, 2003 ; Accepted: March 25, 2003)

ABSTRACT

It is shown that if f is an entire modular integral on $\Gamma(1)$ of weight k , with multiplier system ν , then $f^m(\tau^r)$ is an entire modular integral on $\Gamma(1)$ of weight mk , with multiplier system ν . A more general formula is obtained for the Mellin transforms. Some relations among the Mellin transforms of functions $f(\tau)$, $f^m(\tau^r)$ and $f^m(\tau^r/m', \chi)$ are also deduced.

Key words: Modular integral, Mellin transformation, modular form.

1. INTRODUCTION

Let H be the upper half-plane, Z the set of rational integers, and $\Gamma(1)$ the modular group generated by the matrices $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $\Gamma_0(N)$, for any positive integer N , denote the special congruence subgroup $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : c \equiv 0 \pmod{N} \right\}$. For $\tau \in H$, we have the inversion $\omega(N): \tau \rightarrow -1/(N\tau)$, or as a matrix $\omega(N) = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$ and define $\Gamma_0^*(N)$ to be the group generated by $\Gamma_0(N)$ and $\omega(N)$, [5,7].

In [6], Knopp proved some generalizations of Hecke's celebrated correspondence.

In [9], Weil developed an important generalization to $\Gamma_0^*(N)$ of the Hecke correspondence. In [6] and [7], Knopp deduced the organization of [9], adapting to modular integrals on $\Gamma_0^*(N)$ the arguments that Weil developed for modular forms on $\Gamma_0^*(N)$.

For some results concerning modular forms, we refer to [3,4].

The aim of this paper is to establish a more general formula for the Mellin transforms and to deduce some relations among the Mellin transforms of functions $f(\tau)$, $f^m(\tau')$ and $f^m(\tau'/m', \chi)$

2. MELLIN TRANSFORMS OF MODULAR INTEGRALS ON $\Gamma(1)$

Let f be an entire modular integral (entire MI) on $\Gamma(1)$ of weight k , with multiplier system (MS) ν . That is to say; (i) f satisfies

$$f(\tau+1) = f(\tau), \quad \tau^{-k} f\left(-\frac{1}{\tau}\right) = \nu(V) f(\tau) + q(\tau) \quad (1)$$

(ii). f is holomorphic in H ; (iii). f has a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau} \quad (2)$$

where $k \in 2\mathbb{Z}$ and $q(\tau)$ is a rational period function (RPF). It follows from these three conditions that $a_n = O(n^\gamma)$, $n \rightarrow +\infty$, for some $\gamma > 0$, and this in turn guarantees the

absolute convergence of the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ in the half-plane $\text{Re}(s) > 1 + \gamma$.

This series arises naturally from term-by-term integration when one forms the Mellin transform

$$\Phi_f(s) = \int_0^{\infty} \{f(iy) - a_0\} y^s \frac{dy}{y} = (2\pi)^{-s} \Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s}, \quad (3)$$

of $f(\tau) - a_0$. Note that $\Phi_f(s)$, like the Dirichlet series, is holomorphic in $\text{Re}(s) > 1 + \gamma$

The classic works of Hecke [1,2] show that if f is an entire modular form (that is, if $q=0$ in (1)), then $\Phi_f(s)$ has certain desirable properties, the most striking among them being the functional equation

$$\Phi_f(k-s) = (-1)^{k/2} \Phi_f(s).$$

The Mellin transform of an entire MI on $\Gamma(1)$ with RPF q has precisely the same functional equation as does the Mellin transform of an entire modular form on $\Gamma(1)$.

In addition to $\Phi_f(s)$ we consider the "twisted" functions, introduced by Weil

$$\begin{aligned}
 \text{(i). } f(\tau, \chi) &= \sum_{n=1}^{\infty} a_n \chi(n) e^{2\pi i n \tau} \\
 \text{(ii). } \Phi_f(s, \chi) &= \left(\frac{m'}{2\pi}\right)^s \Gamma(s) \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}
 \end{aligned} \tag{4}$$

related, respectively, to f and Φ_f . Here, χ is a primitive character modulo $m' \in \mathbb{Z}^+$, $(m', N) = 1$. Note that $\Phi_f(s, \chi)$ is the Mellin transform of $f(\tau/m', \chi)$. [6].

3. MAIN RESULTS

Theorem 1: If $f(\tau)$ is an entire MI on $\Gamma(1)$ of weight k , with MS υ , then $f^m(\tau^r)$ is an entire MI on $\Gamma(1)$ of weight mk , with MS υ , for all $m, r \in \mathbb{Z}^+$ and $\tau, \tau^r \in H$.

Proof: Let $f(\tau)$ be an entire MI on $\Gamma(1)$ of weight k , with MS υ . That is, f is holomorphic in H and satisfies equations (1). Also, f has a Fourier expansion of the form (2). $f^m(\tau^r)$ is well defined. Further, we obtain,

$$f^m(\tau^r + 1) = f^m(\tau^r)$$

and

$$\tau^{-mk} f^m(-1/\tau^r) = \upsilon(V) f^m(\tau^r) + q(\tau^r)$$

Also $f^m(\tau^r)$ has the Fourier expansion of the form

$$f^m(\tau^r) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \xi} \quad , \quad \text{for } \xi = \tau^r \tag{5}$$

$f^m(\tau^r)$ is holomorphic in H for $\tau^r \in H$. Hence, $f^m(\tau^r)$ is an entire MI on $\Gamma(1)$ of weight mk , with MS υ . As in [8], any RPF on $\Gamma(1)$ with poles in Q has the form

$$q(\tau^r) = \sum \alpha_1 \tau^{-r1} \quad , \quad -S \leq 1 \leq K, \tag{6}$$

Corollary 1: For $m=1$, $f(\tau^r)$ is an entire MI on $\Gamma(1)$ of weight k , with MS υ .

Theorem 2: Let f and $f^m(\tau^r)$ be as in Theorem 1. The Mellin transform of the function $f^m(\tau^r)$ is

$$\Phi_f(s) = \frac{1}{r} (i)^{\frac{s}{r} - s} (2\pi)^{-s/r} \Gamma(s/r) \sum a_n n^{-s/r} \tag{7}$$

for $m, r \in \mathbb{Z}^+$, $\tau^r \in H$. Also we have,

$$\Phi(mk-s) = (-1)^{mk/2} \Phi(s).$$

Proof: From the Fourier expansion (5) and by the definition of Mellin transform, we have, for $\zeta = (iy)^r$

$$\begin{aligned} \Phi(s) &= \int_0^{\infty} \{ f^m((iy)^r) - a_0 \} y^s \frac{dy}{y} = \sum_{n=1}^{\infty} \int_0^{\infty} a_n e^{2\pi i n \zeta} y^s \frac{dy}{y} \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} a_n e^{-u} \left(\frac{u}{2\pi n} \right)^{s/r} \left(\frac{1}{r} \right) (i)^{\frac{s}{r}-s} \frac{du}{u}. \end{aligned}$$

Putting $iy = (i \frac{u}{2\pi n})^{1/r}$, it is obtained

$$\Phi(s) = \frac{1}{r} (i)^{\frac{s}{r}-s} (2\pi)^{-s/r} \Gamma(s/r) \sum a_n n^{-s/r}.$$

where $\Gamma(s) = \int_0^{\infty} e^{-t} t^s \frac{dt}{t}$. Further, it is easily verified that,

$$\Phi(mk-s) = (-1)^{mk/2} \Phi(s).$$

Theorem 3: Let f and $f^m(\tau^r)$ be as in Theorem 1, Φ_f the Mellin transform of $f(\tau) - a_0$, defined by (3), and Φ the Mellin transform of $f^m(\tau^r) - a_0$, defined by (7). Then, we have,

$$\Phi(s) = \left(\frac{1}{r} \right) (i)^{\frac{s}{r}-s} \Phi_f(s/r).$$

Proof: It is easily verified from the equations (3) and (7).

Theorem 4: Let f and $f^m(\tau^r)$ be as in Theorem 1. Let χ be a primitive Dirichlet character modulo m' . Then the Mellin transform of $f^m(\tau^r/m', \chi)$ is

$$\Phi(s, \chi) = \frac{1}{r} (i)^{\frac{s}{r}-s} (2\pi)^{-s/r} (m')^{s/r} \Gamma(s/r) \sum_{n=1}^{\infty} a_n \chi(n) n^{-s/r}. \quad (8)$$

for $m, r \in \mathbb{Z}^+$, $\tau^r \in H$. Also we have,

$$\Phi(s, \chi) = \frac{1}{r} (i)^{\frac{s}{r}-s} \Phi_f(s/r, \chi) \quad (9)$$

Proof: Putting $iy = (i \frac{um'}{2\pi m})^{1/r}$ and using the Mellin transform, (8) is proved. The proof is similar to that of Theorem 2. The functional equation (9) is easily satisfied from the equations (4) and (8).

Corollary 2 : a. If we put $r=1$ in (7), we get equation (3).
 b. If we put $r=1$ in (8), we get equation (4).

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