Commun. Fac. Sci. Univ. Ank. Series A1
V. 49. pp. 131 - 137 (2000)

DISCRETE SETS AND IDEALS

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(Received Oct. 17, 2000; Accepted Dec. 12, 2000)

ABSTRACT

In this paper, the discrete sets and corresponding dual ideals and principal maximal ideals in B(X) are studied, where X is an n-dimensional complex manifold and B(X) is a ring (algebra) of holomorphic functions defined on X.

1. INTRODUCTION

a) Let us denote the open unit disc in C by U and the unit disc bounding U by T. Similary, in C^n, the open unit disc and its boundary are defined by

$$U^n = \{ z \in C^n : | z_i | < 1, \ 1 \leq i \leq n \}$$

and

$$T^n = \{ z \in C^n : | z_i | = 1, \ 1 \leq i \leq n \}$$

respectively.

U^n is the cartesian product of U by itself n times and T^n is the cartesian product of T by itself n times. For $n > 1$, T^n is a subset of the topological boundary ∂U^n. If n=1, then U^1=U and $T^1 = \partial T$.

b) More generally, an open polydisc in C^n is the cartesian product of n open discs. The polydisc with radius r = ($r_1, r_2, \ldots, r_n$) and center $z^0 = (z_1^0, z_2^0, \ldots, z_n^0)$ is

$$P_r^n = \{ z \in C^n : |z_i - z_i^0| < r_i, \ 1 \leq i \leq n \}$$

and the boundary of P_r^n is defined by

$$T_r^n = \{z \in C^n : |z_i - z_i^0| = r_i, 1 \le i \le n\}$$

The closure of U^n defined by $\overline{U}^n$. Then $\overline{U}^n = U^n \cup T^n$. i.e.

$$\overline{U}^n = \{z \in C : |z_i - z_i^0| \le 1, 1 \le i \le n\}$$

The problem of discarding the slower is of great importance in practice, [6].

1.1. Definition. Let X be a topological space and let $D \subset X$. If D has no limit points, then it is called a discrete subset (of X)

Let G be a region (open connected set) in C, and let A(G) be the ring (or complex alcebra) of complex valued analytic functions in G. The set of zeros of f in G, $S(f)=\{z \in G: f(z)=0\}$ for $f \in A(G)$, is a discrete set.

Here S(f) is thought algebraically. That is, the zeros are counted by multiplicity in S(f) and also in the union and intersection. If K is a subset of A(G), then $S(K)= \bigcup_{f \in K} S(f)$. The following lemmas are well-known from [3].

1.2. Lemma. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a discrete sequence, $\{m_k\}$ be a discrete sequence of positive integers and $\{\beta_{k,p} : p = 0,1,......,m_{k-1} : k = 1,2,......\}$ be a sequence of complex numbers. Then there exists an $f \in A(G)$ so that $f^{(p)}(\alpha_k) = \beta_{k,p}$. $(p = 0,1,..., m_{k-1} : k = 1,2,..)$.

1.3. Lemma. Let $f_1, f_2 \in A(G)$ and let $S(f_1) \cap S(f_2) = \phi$. Then for every $h \in A(G)$, there exist $g_1, g_2 \in A(G)$ so that $h = f_1 g_1 + f_2 g_2$.

1.4. Lemma. If $f_1, f_2 \in A(G)$, then there exists $g_1, g_2 \in A(G)$ so that $S(f_1 g_1 + f_2 g_2) = S(f_1) \cap S(f_2)$.

2. DUAL IDEALS

Let I be an ideal of A(G). If there exists a point $z_0 \in G$ so that $f(z_0)=0$ for every $f \in I$, then I is called an ideal of type I, and in general it is denoted by I_{z_0}. Then

$$I_{z_0} = \{f \in A(G): f(z_0)=0\}$$

Other ideals of A(G) are called of type II.

2.1. Definition. Let us denote a family of nonempty discrete subsets of G by H. If the following conditions are satisfied, then H is called the dual ideal (of G).

1) If $D_1, D_2 \in H$ then $D_1 \cap D_2 \in H$
2) If $D_1 \in H$ and D_2 is a discrete subset of G such that $D_1 \subset D_2$, than $D_2 \in H$.

By Zorn lemma there exists a maximal dual ideal. (Let B be a dual ideal of G. If there is not a dual ideal B^1 of B so that B^1 contains B as a proper subset then B is called maximal dual ideal.) If B is a maximal dual ideal, then there exists a discrete set $D \in H$ such that $D \cap D^1 = \phi$ for every discrete subset D^1 not belonging to H.

Let B be the maximal dual ideal of discrete subsets of G. If there exists a point $z_0 \in G$ such that $z_0 \in D$ for every $D \in H$ then B is called a maximal dual ideal of type I. All other maximal dual ideals of discrete subsets of G are called maximal dual ideals of type II.

2.2. Theorem. 1) For every maximal dual ideal B of discrete subsets of G $I(B)=\{f: f \in A(G), S(f) \in B\}$ is a maximal dual ideal of A(G).

2) Conversely, for every maximal ideal I of A(G), $B(I)=\{S(f): f \in I\}$ is a maximal dual ideal of discrete subsets of G.

3) Let us denote the set of maximal ideals of A(G) by M and the set of maximal dual ideals of discrete subsets of G by N. Then the maps ϕ and ψ defined by $\phi: N \rightarrow M$, $\phi(B)=I(B)$ and $\psi: M \rightarrow N$, $\psi(I(B))=B$ are one to one and onto. B is a maximal dual ideal of type I or II according as the corresponding I(B) is a maximal ideal of type I or II [3].

2.3. Theorem. Let R be an open Riemann surface, A(R) be ring of analytic functions defined on R and B be a dual ideal of R then $I(B)=\{f \in A(R): S(f) \in B\}$ is an ideal of A(R).

Proof. If $f_1, f_2 \in I(B)$ then $S(f_1)$, $S(f_2) \in B$. Since B is a dual ideal $S(f_1) \cap S(f_2) \in B$. As $S(f_1) \cap S(f_2) \subset S(f_1 - f_2)$, $S(f_1 - f_2) \in B$ and therefore $f_1 - f_2 \in I(B)$.
Let $f \in I(B)$ and $g \in A(R)$ be arbitrary. As $S(f) \in B$ and $S(f) \subset S(fg)$ we have $S(fg) \in B$. Then $fg \in I(B)$ and therefore I(B) is an ideal of A(R). Also if $B_1 \subset B_2$ then $I(B_1) \subset I(B_2)$ is obvious.

2.4. Theorem. $A_D^1 = \{f \in A(G): \text{ for every } z \in D, f'(z) = 0\}$ is a subring of A(G) for a discrete subset D of G. (Here f' denotes the derivative of f)

Proof. If f,g$\in A_D^1$ then as $(f-g)'(z)=(f'-g')(z)=0$ for every z$\in$D, f-g$\in A_D^1$. Similary as $(fg)'(z)=0$ for every z$\in$D, A_D^1 is a subring of A(G).

Corollary. If $A_D^{(n)}=\{g\in A_D^{(n-1)}: g^{(n)}(z)=0 \; z\in D, n\geq 2\}$ then $A_D^{(n)}$ is a subring of $A_D^{(n-1)}$. Further $\bigcap_{N=1}^{\infty} A_D^{(n)}=C$.

Proof. If f$\in \bigcap_{N=1}^{\infty} A_D^{(n)}$ then $f^{(n)}(z)=0$ for n=1,2,.... (z$\in$D) This implies that f is a constant.

3. COVERING SPACES

3.1. Definition. Let X and $\tilde{X}$ be two topological spaces and let p: $\tilde{X}\to X$ be a continuous map. If the following conditions are satisfied then $\tilde{X}$ is called the covering space of X.

1) For every x$\in$X, there exists an open neighbourhood W of x so that $p^{-1}(W)$ is union of some open sets W_α in $\tilde{X}$ ($\alpha\in I$).
2) $p|W_\alpha$ is a local homeomorphism of W_α onto W ($\alpha\in I$).

If $\tilde{X}$ is a covering space of X, the map p is called a covering map. If p($\tilde{X}$)=X then X is called the projection of $\tilde{X}$.

3.2. Definition. Let $\tilde{X}$ be a covering space of X, p: $\tilde{X}\to X$ a covering map and g: $\tilde{X}\to\tilde{X}$ be a homeomorphism. If pog = p i.e. $p(g(\tilde{x}))=p(\tilde{x})$ then g is called a covering map of $\tilde{X}$.

Hence a covering map permutes the points with the same projections. The covering transformations form a group under combination. This group is called the group of covering transformations, [2], [4].

Let p: $\tilde{X}\to X$ be a covering map and x$\in$X where X is a Hausdorff space. Let W be a neighbourhood of x in the meaning of Definition 3.1. Let us take a neighbourhood U of x so that $\bar{U}\subset W$. If we form a set K=$\{k_\alpha\}$ for each W_α where $k_\alpha\in(W_\alpha\cap p^{-1}(U))$ then the following lemma can be given.

3.3. Lemma. K is a discrete set.

Proof. Conversely let us suppose k is a limit point of K. Let V be a neighbourhood of p(k). Since p is continuous, there exists a neighbourhood V_1 of k so that $p(V_1)\subset V$. Let $k_\alpha\in(V_1\text{-}k)\cap K$ then $p(k_\alpha)\in U$. Hence $V\cap U\neq\phi$. That is the intersection of a neighbourhood of p(k) with U is nonempty. Hence p(k) is a limit point of U. That is $p(k)\in\overline{U}$. Since $\overline{U}\subset W$, there exists a W_α so that $k\in W_\alpha$. But there can only be k_α in W_α by hypothesis. Therefore k can not be a limit point of K

Notice that if $\tilde{X}$ is a covering space of X and p: $\tilde{X}\to X$ is a covering map then $p^{-1}(x)$ has a discrete topology for every $x\in X$. Because the intersection of the open set W_α with $p^{-1}(x)$ consist of one point. Therefore this point is open in the subspace topology on $p^{-1}(x)$. Further for $x,y\in X$ the cardinalities of $p^{-1}(x)$ and $p^{-1}(y)$ are equal.

3.4. Definition. Let R be a Riemann surface and D be a discrete subset of R. The ideal $I_D=\{f\in A(R)\text{: } f(p)=0, \text{ for } p\in D\}$ is called discrete ideal of A(R). For $I_q=\{ f\in A(R)\text{: } f(q)=0\}$ we can give the following theorem.

3.5. Theorem. Let R and $\tilde{R}$ be two Riemann surfaces, $\tilde{R}$ be a covering surface of R, $p:\tilde{R}\to R$ be a covering map and g: $\tilde{R}\to\tilde{R}$ be a covering transformation. Then

a) Let $A=\{ I_{q_i} : q_i\in p^{-1}(x)\}$ for $x\in R$. Then the map $\phi : A\to A$, $\phi(q_i) = I_{g(q_i)}$ is one-to-one and onto.

b) Let $B=\{ I_{p^{-1}(x)} : x\in R\}$. Then $\psi : R\to B$, $\psi(x) = I_{p^{-1}(x)}$ is one-to-one and onto.

Proof. a) First we show that ϕ is a map. If $I_{q_1} = \{f\in A(\tilde{R}): f(q_1)=0\} = I_{q_2} = \{g\in A(\tilde{R}): g(q_2)=0\}$ then there exists $f\in I_{q_1}$ so that $S(f)=\{q_1\}$ by [1] and $I_{q_1} = \langle f\rangle = \{gf : g\in A(\tilde{R})\}$. Since $f\in I_{q_2}$, $f(q_2)=0$. Then $q_1=q_2$. Therefore since $g(q_1)=g(q_2)$, $\phi(I_{q_1})=\phi(I_{q_2})$. That is ϕ is a map. If $\phi(I_{q_1})=\phi(I_{q_2})$, then $I_{g(q_1)} = I_{g(q_2)} \Rightarrow g(q_1)=g(q_2) \Rightarrow q_1=q_2 \Rightarrow I_{q_1} = I_{q_2}$, i.e. ϕ is one-to-one. Finally let $I_{q_1}\in A$. Since g is onto there exists a $q_j\in p^{-1}(x)$ so that $g(q_j)=q_i$. Then $\phi(I_{q_j}) = I_{q_i}$

b) It is easy to see that ψ is a map. To show that it is one-to-one let $\psi(x)=\psi(y)$, i.e. , $I_{p^{-1}(x)} = I_{p^{-1}(y)}$. Then since $p^{-1}(x)$ is a discrete set, by generalized Weierstrass theorem there exists a $f\in A(R)$ so that $S(f)= p^{-1}(x)$ [5]. But since $f\in I_{p^{-1}(y)}$, $S(f)=p^{-1}(y)$. Let $x_i=y_i$ where $x_i\in p^{-1}(x)$ and $y_i\in p^{-1}(x)$. Then $x=p(x_i)=p(y_i)=y$. This shows that ψ is one-to-one. By the definition ψ is onto.

4. n- DIMENSIONAL COMPLEX MANIFOLDS

4.1. Definition. . Let X be a topological space, U be an open subset of X, and ψ be a topological map from U to C^n. The pair (U, ψ) is called coordinate card or card in X. If a $\in$U then (U, ψ) is said to contain a.

4.2. Definition. Let X be a connected Hausdorff space and $\phi = \{(U_i, \psi_i) : i \in I\}$ be set of cards in X. If the following conditions are satisfied then X=(X,ϕ) is called an n-Dimensional Complex Manifold.

1) Every x$\in$X is in only one card. That is the family $\{U_i: i\in I\}$ forms an open cover of X
2) If (U_1, ψ_1), $(U_2, \psi_2) \in \phi$ and $U_1 \cap U_2 \neq \phi$ then

$$\psi_{12} = \psi_1 o \psi_2^{-1} : \psi_2(U_1 \cap U_2) \to \psi_1(U_1 \cap U_2)$$

is a topological map.

When ψ_{12} is analytic, the manifold X=(X,ϕ) is called n- Dimensional Analytic Manifold. Here the family ϕ is called an analytic structure (or atlas) on X. Every $x \in U_i$ is determined uniquely by $\psi_i(x)$. These ψ_i' s are called local parameters or local variables, [7].

Let X=(X, ϕ) be an analytic manifold and W$\subset$X be an open set. Further suppose that $x_0 \in W$ and f is a complex valued function on W. If there exists a neighbourhood $U_{(x_0)}$ of x_0 so that $U_{(x_0)} \subset W \cap U_i$ where $f o \psi_i^{-1}$ is holomorphic in $\psi_i(U_i) \subset B_i$, then f is called holomorphic at x_0. (B_i is an open set in C^n) If f is holomorphic at every point of W then f is called holomorphic on W. In particular if W=X then f is holomorphic on X.

4.3. Theorem. Let X be an analytic manifold of dimension n and B(X) be a ring of bounded, holomorphic functions (or complex algebra) defined on X. Also suppose that

1) For every x$\in$X there exists an f$\in$B(X) having a simple zero at x and no other zeros.
2) For every discrete sequence (x_n) in X there exists f$\in$B(X) so that lim $f(x_n)$ does not exist.

Then the necessary and sufficient contition for a maximal ideal in B(X) to be essential is that it is of the first type.

Proof. First we suppose that I$\in$B(X) is essential, i.e. $I = \langle f \rangle = \{gf : g \in B(X)\}$. f has a zero. Then inf $\{|f(x)| : x\in X\} = 0$. In this case there exists a sequence (x_n) in

X so that lim $f(x_n)=0$. If $g\in I$ then there exists $h\in B(X)$ so that $g=fh$. Since h is bounded lim $g(x_n)=0$. Then for every $g\in B(X)$ lim $g(x_n)$ exists. By hypothesis (x_n) can not be discrete. That is $x_n\to x\in X$. Therefore the necessary and sufficient condition for $g\in B(X)$ to be $g\in I=\langle f\rangle$ is that $g(x)=0$, i.e. $I=I_x$

Conversely let $I\in B(X)$ be of the first type, i.e. $I=I_{x_0}=\{f\in B(X)\colon f(x_0)=0\}$ then by hypothesis there exists an $f\in B(X)$ having a simple zero at x_0 but no other zeros. Now let us think the essential ideal $\langle f\rangle$. It is clear that f is a proper ideal. If ϕ: $B(X)\to C$, $\phi(g)=g(x_0)$ is defined then the kernel of ϕ is $\langle f\rangle$ and the ideal $\langle f\rangle$ is maximal. But as I_{x_0} is maximal, $I_{x_0}=\langle f\rangle$. That is the first type maximal ideal of B(X) is essential maximal ideal.

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