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IDEALIZATION OF PRZEMSKI'S DECOMPOSITION THEOREMS

E. HATIR

Dept. of Math., Selçuk University, Education Faculty, 42090, Konya, TURKEY
e-mail: ehatir@selcuk.edu.tr

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ABSTRACT

Przemski [5] introduced D(c,α)-set, D(c,p)-set, D(c,s)-set and D(α,p)-set to obtain several decompositions of continuity. In this paper, we extend these sets and obtain new decompositions of continuity via idealization.

Key words and phrases. D(c,α)-set, D(c,p)-set, D(c,s)-set, D(c,α)-continuity, D(c,p)-continuity, D(c,s)-continuity

1. INTRODUCTION AND PRELIMINARIES

In 1993, Przemski [5] introduced D(c,α)-set, D(c,p)-set, D(c,s)-set and D(α,p)-set to obtain several decompositions of continuity. In [4], the authors introduce the notion of α-I-open, semi-I-open and β-I-open sets via ideal. In this paper, we extend Przemski's sets and obtain decompositions of continuity via idealization.

Throughout this paper, for a subset A of a topological space (X,τ), Cl(A) and Int(A) denote the closure of A and the interior of A, respectively. Let (X,τ) be a topological space and I an ideal of subsets of X. An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions: (1) if $A\in I$ and $B\subset A$, then $B\in I$; (2) if $A\in I$ and $B\in I$, then $A\cup B\in I$. An ideal topological space, denoted by (X,τ,I), is a topological space (X,τ) with an ideal I on X. For a subset $A\subset X$, $A^*(\tau,I)=\{x\in X : U\cap A\notin I$ for each neighborhood U of x$\}$ is called the local function [1] of A with respect to I and τ. We simply write A* instead of A*(τ,I) in case there is no chance for confusion. X* is often a proper subset of X. The hypothesis X=X* [9] is equivalent to the hypothesis $\tau\cap I=\varnothing$ [9]. For every ideal topological space (X,τ,I), there exists a topology τ*(I), finer than τ, generated by the base $\beta(I,\tau)=\{U\setminus I : U\in\tau \text{ and } I\in I\}$. However, β(I,τ) is not always a topology [9]. It is well known that $Cl^*(A)=A\cup A^*$ defines a Kuratowski closure operator for τ*(I). In 1992, Janković and Hamlett [3] introduced the notion of I-open sets in ideal topological spaces. For $A\subset(X,\tau,I)$, A is said to be I-open if $A\subset Int(A^*)$. In 1999, Dontchev [6] has introduced the notion of pre-I-open which is weaker than I-open. For $A\subset(X,\tau,I)$, A is said to be pre-I-open if $A\subset Int(Cl^*(A))$. Quite recently, Hatir and Noiri [4] have introduced the notion of α-I-open (resp. semi-I-open, β-I-open) as in the following; A subset A of a space (X,τ,I) is said to be α-I-open (resp. semi-I-open, β-I-open) if $A\subset Int(Cl^*(Int(A)))$ (resp. $A\subset Cl^*(Int(A))$, $A\subset Cl(Int(Cl^*(A)))$).

For a topological space (X,τ), Przemski [5] defined the following sets;
a) $D(c,\alpha)=\{B\subset X : Int(B)=B\cap Int(Cl(Int(B)))\}$,
b) $D(c,p)=\{B\subset X : Int(B)=B\cap Int(Cl(B))\}$,
c) $D(c,s)=\{B\subset X : Int(B)=B\cap Cl(Int(B))\}$,
d) $D(\alpha,p)=\{B\subset X : B\cap Int(Cl(Int(B)))= B\cap Int(Cl(B))\}$,
e) $D(c,ps)= \{B\subset X : Int(B)=B\cap Cl(Int(Cl(B)))\}$ (Dontchev and Przemski [7]).

2. $D_I(c,p)$-set, $D_I(c,\alpha)$-set, $D_I(c,s)$-set, $D_I(c,ps)$-set, $D_I(\alpha,p)$-set

Now, we can give the following definition via ideal.

Definition 2.1. For an ideal topological space (X,τ,I), we define the following;
a) $D_I(c,p)=\{A\subset(X,\tau,I) : Int(A)= A\cap Int(Cl^*(A))\}$,
b) $D_I(c,\alpha)=\{A\subset(X,\tau,I) : Int(A)= A\cap Int(Cl^*(Int(A)))\}$,
c) $D_I(c,s)=\{A\subset(X,\tau,I) : Int(A)= A\cap Cl^*(Int(A))\}$,
d) $D_I(c,ps)=\{A\subset(X,\tau,I) : Int(A)= A\cap Cl(Int(Cl^*(A)))\}$,
e) $D_I(\alpha,p)=\{A\subset(X,\tau,I) : A\cap Int(Cl^*(Int(A)))=A\cap Int(Cl^*(A))\}$.

Proposition 2.1. The following statements hold for an ideal topological space (X,τ,I),
a) Every $D(c,p)$-set is $D_I(c,p)$-set,,
b) Every $D(c,\alpha)$-set is $D_I(c,\alpha)$-set,
c) Every $D(c,s)$-set is $D_I(c,s)$-set,
d) Every $D(c,ps)$-set is $D_I(c,ps)$-set.

Proof. a) Let A be a $D(c,p)$-set. Since $A\subset A^*\cup A=Cl^*(A)$,

$$Int(A)\subset A\cap Int(Cl^*(A))=A\cap Int(A^*\cup A)\subset A\cap Int(Cl(A)\cup A)=A\cap Int(Cl(A))=Int(A).$$

Therefore, A is $D_I(c,p)$-set.

b) Let A be a $D(c,\alpha)$-set. Then

$$Int(A)\subset A\cap Int(Cl^*(Int(A)))=A\cap Int((Int(A))^*\cup Int(A))\subset A\cap Int(Cl(Int(A))\cup Int(A)) =A\cap Int(Cl(Int(A)))=Int(A).$$

This shows that A is $D_I(c,\alpha)$-set.

c) Let A be a $D(c,s)$-set. Then

$$Int(A)\subset A\cap Cl^*(Int(A))=A\cap((Int(A))^*\cup Int(A))\subset A\cap(Cl(Int(A)\cup Int(A))=A\cap Cl(Int(A))=Int(A).$$

This shows that A is $D_I(c,s)$-set.

d) Let A be a D(c,ps)-set. Then

$$\mathrm{Int}(A)\subset A\cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A)))=A\cap \mathrm{Cl}(\mathrm{Int}(A^*\cup A))\subset A\cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)\cup A))$$
$$=A\cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))=\mathrm{Int}(A).$$

This shows that A is D_I(c,ps)-set.

Remark 2.1. None of them in the Proposition 2.1 is reversible as shown by examples below.

Example 2.1. Let X={a,b,c,d}, τ={∅, X, {a}, {b,d}, {a,b,d}}and I={∅, {a}, {b},{a, b}}. Take A={a,b}. Since A*={a,b}*=∅ and Int(Cl*(A))={a}, A is D_I(c,p)-set and D_I(c,ps)-set, but not D(c,p)-set and D(c,ps)-set.

Example 2.2. Let X={a,b,c,d}, τ={∅, X, {d}, {a,c}, {a,c,d}}and I={∅, {c}, {d},{c, d}}. Put A={b,d}. Since (Int(A))*=({d})*=∅ and Cl*(Int(A))={d}, A is D_I(c,s)-set, but not D(c,s)-set.

Example 2.3. Let X={a,b,c,d}, τ={∅, X, {c}, {b,c}, {a,b,c}, {a,c,d}, {a,c}} and I={∅, {c}, {d},{c,d}}. Take A={c,d}. Since (Int(A))*=({c})*=∅ and

$$\mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A)))=\{c\},$$

A is $D_I(c,\alpha)$-set, but not $D(c,\alpha)$-set. Furthermore, A is not D_I(c,ps)-set.

Example 2.4. Let X={a,b,c,d}, τ={∅, X, {d}, {a,d}, {b,c,d}}and I={∅, {c}, {d},{c, d}}. Take A={b,d}. Since (Int(A))*=({d})*=∅ and Int(Cl*(Int(A)))={d}, A is $D_I(c,\alpha)$-set, but not D_I(c,p)-set.

Example 2.5. Let X={a,b,c,d}, τ={∅, X, {b}, {c,d}, {b,c,d}}and I={∅, {c}}. Then A={a,b} is $D_I(c,\alpha)$-set, but not D_I(c,s)-set. Because

$$\mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A)))=\mathrm{Int}(\mathrm{Cl}^*(\{b\}))=\{b\},$$

therefore A is $D_I(c,\alpha)$-set. Since Cl*(Int(A))=({b})*={a,b}, A is not D_I(c,s)-set.

Proposition 2.2. The following statements hold for an ideal topological space (X,τ,I),

a) Every D_I(c,ps)-set is D_I(c,p)-set,,

b) Every D_I(c,p)-set is $D_I(c,\alpha)$-set,

c) Every D_I(c,s)-set is $D_I(c,\alpha)$-set.

Proof. a) Let A be a D_I(c,ps)-set. Since

$$\mathrm{Int}(A)=A\cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A)))\supset A\cap \mathrm{Int}(\mathrm{Cl}^*(A))\supset \mathrm{Int}(A),$$

we have Int(A)=A∩Int(Cl*(A)). This shows that A is D_I(c,p)-set.

b) Let A be a D_I(c,p)-set. Since

$$\mathrm{Int}(A)=A\cap \mathrm{Int}(\mathrm{Cl}^*(A))\supset A\cap \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A)))\supset \mathrm{Int}(A),$$

we have Int(A)=A∩Int(Cl*(Int(A))). This shows that A is $D_I(c,\alpha)$-set.

c) Let A be a D_I(c,s)-set. Since

$$\text{Int}(A)=A\cap \text{Cl}^*(\text{Int}(A))\supset A\cap \text{Int}(\text{Cl}^*(\text{Int}(A)))\supset \text{Int}(A),$$

we have $\text{Int}(A)=A\cap \text{Int}(\text{Cl}^*(\text{Int}(A)))$. This shows that A is $D_I(c,\alpha)$-set.

According to the Examples and Proposition 2.2, we can give the following diagram:

$$\begin{array}{ccccc} D_I(c,ps)\text{-set} & \Rightarrow & D_I(c,p)\text{-set} & \Rightarrow & D_I(c,\alpha)\text{-set} \\ & & & & \Uparrow \\ & & & & D_I(c,s)\text{-set} \end{array}$$

Remark 2.2. a) Every open set is α-I-open, pre-I-open, semi-I-open and β-I-open, but not conversely (Remark 2.1 in [4]).
b) Every open set is $D_I(c,p)$-set, $D_I(c,\alpha)$-set, $D_I(c,s)$-set and $D_I(c,ps)$-set.
This fact follows from Proposition 2.1 since every open set is $D(c,\alpha)$-set, $D(c,p)$-set, $D(c,s)$-set and $D(c,ps)$-set.

Proposition 2.3. For an ideal topological space (X,τ,I), $A\subset X$ and $I=\varnothing$, we have
a) A is $D_I(c,p)$-set if and only if A is $D(c,p)$,
b) A is $D_I(c,\alpha)$-set if and only if A is $D(c,\alpha)$-set,
c) A is $D_I(c,s)$-set if and only if A is $D(c,s)$-set,
d) A is $D_I(c,ps)$-set if and only if A is $D(c,ps)$-set.
Proof. Sufficiency is shown in Proposition 2.1. For necessity note that in case of the minimal ideal $A^*=\text{Cl}(A)$.

Theorem 2.1. For an ideal topological space (X,τ,I), the following hold;
a) A is an open if and only if it is both α-I-open and $D_I(c,\alpha)$-set,
b) A is an open if and only if it is both pre-I-open and $D_I(c,p)$-set,
c) A is an open if and only if it is both semi-I-open and $D_I(c,s)$-set,
d) A is an open if and only if it is both β-I-open and $D_I(c,ps)$-set.
Proof. Remark 2.2 gives the necessity.

For the sufficiency of (a); Suppose that A is both α-I-open and $D_I(c,\alpha)$-set. Then $A\subset \text{Int}(\text{Cl}^*(\text{Int}(A)))$ and $\text{Int}(A)=A\cap \text{Int}(\text{Cl}^*(\text{Int}(A)))$. Since

$$A\subset A\cap \text{Int}(\text{Cl}^*(\text{Int}(A)))=\text{Int}(A),$$

A is an open set.

(b), (c) and (d) are analogous with the Proof of (a) and are thus omitted.

3. DECOMPOSITIONS OF CONTINUITY

Definition 3.1. A function $f:(X,\tau,I)\rightarrow(Y,\sigma)$ is said to be pre-I-continuous [6] (resp. α-I-continuous [4], semi-I-continuous [4], β-I-continuous [4]) if for each open set V of (Y,σ), the set $f^{-1}(V)$ is pre-I-open (resp. α-I-open, semi-I-open, β-I-open)

Definition 3.2. A function $f:(X,\tau,I)\rightarrow(Y,\sigma)$ is said to be $D_I(c,\alpha)$-continuous (resp. $D_I(c,p)$- continuous, $D_I(c,s)$-continuous, $D_I(c,ps)$-continuous, $D_I(\alpha,p)$-continuous) if for each open set V of (Y,σ), the set $f^{-1}(V)$ is a $D_I(c,\alpha)$-set (resp. $D_I(c,p)$-set, $D_I(c,s)$-set, $D_I(c,ps)$-set, $D_I(\alpha,p)$-set).

By PIO(X) (resp. αIO(X)), we denote the family of all pre-I-open (resp. α-I-open) sets of a space (X,τ,I).

Theorem 3.1. $PIO(X)\cap D_I(\alpha,p)=\alpha IO(X)$.
Proof. Let $B\in PIO(X)$ and $B\in D_I(\alpha,p)$. Then $B\subset Int(Cl^*(B))$ and $B\cap Int(Cl^*(B))=B\cap Int(Cl^*(Int(B)))$. Since $B\subset Int(Cl^*(B))$, we have $B\subset Int(Cl^*(B))\cap B=B\cap Int(Cl^*(Int(B)))\subset Int(Cl^*(Int(B)))$, that is, B is an α-I-open.

Conversely, Let B be an α-I-open and since $B\subset Int(Cl^*(Int(B)))\subset Int(Cl^*(B))$, B is pre-I-open. Moreover, $B\subset B\cap Int(Cl^*(Int(B)))\subset B\cap Int(Cl^*(B))\subset B$, that is, $B\in D_I(\alpha,p)$-set.

It is known that α-continuous functions into a regular space are continuous [8]. Since every α-I-open set is an α-open set, the following corollary follows,

Corollary 3.1. If (Y,σ) is a regular space, $f:(X,\tau,I)\rightarrow(Y,\sigma)$ is continuous if and only if it is both pre-I-continuous and $D_I(\alpha,p)$-continuous.
Proof. It follows from Theorem 3.1 and since every α-I-open set is an α-open [4].

From Theorem 2.1 we can give the following theorem,

Theorem 3.2. For an ideal topological space (X,τ,I), the following statement are equivalent
a) f is continuous,
b) f is α-I-continuous and $D_I(c,\alpha)$-continuous,
c) f is pre-I-continuous and $D_I(c,p)$-continuous,
d) f is semi-I-continuous and $D_I(c,s)$-continuous,
e) f is β-I-continuous and $D_I(c,ps)$-continuous.

Recall the following definition in [4].

Definition 3.3. A subset A of an ideal topological space (X,τ,I) is called
a) t-I-set if $Int(Cl^*(A))=Int(A)$,
b) α^*-I-set if $Int(Cl^*(Int(A)))=Int(A)$,
c) S-I-set if $Cl^*(Int(A))=Int(A)$.

Note that every t-I-set is $D_I(c,p)$-set, every S-I-set is $D_I(c,s)$-set and every α^*-I-set is $D_I(c,\alpha)$-set, but not conversely, see the following example.

Example 3.1. Let X={a,b,c,d}, τ={∅, X, {b}, {d}, {b,d}, {a,b}, {c,d}, {b,c,d}, {a,b,d}} and I={∅, {a}, {c},{a,c}}. Then A={b,c} is $D_I(c,p)$-set, $D_I(c,s)$-set and $D_I(c,\alpha)$-set, but it is not t-I-set, S-I-set and α^*-I-set.

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