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ON APPROXIMATION AND INTERPOLATION OF ENTIRE FUNCTIONS IN TWO COMPLEX VARIABLES WITH INDEX-PAIR (p,q)

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ABSTRACT

The present paper deals with the characterization of the (p,q)-type of entire functions $f : C^2 \to C$ in terms of the Chebyshev best approximation to f on compact set $E \subset C^2$ by polynomials.

1. INTRODUCTION

Let E be a bounded closed set in the space C^2 of two complex variables $z = (z_1, z_2)$, with the norm

$$\|f\|_E = \sup\{|f(z)| : z \in E\}$$

for a function f defined and bounded on E.

Let P_ν denote the set of all polynomials in z of degree $\le \nu$. Set

$$E_\nu(f, E) = \inf\{\|f - p\|_E : p \in P_\nu\}.$$

Winiarski [5] proved the following theorem for one complex variable:

Theorem A. A function f, defined and bounded on a closed set E with a positive transfinite diameter d, can be continued to an entire function f of order $\rho(0 < \rho < \infty)$ and of type $\sigma(0 < \sigma < \infty)$, if and only if

$$\limsup_{\nu \to \infty} \nu^{1/\rho}(E_\nu(f, E))^{1/\nu} = d(e\sigma\rho)^{1/\rho}. \tag{1.1}$$

In two complex variables, the type σ of $f(z)$ can not be characterized by means of the measure of the Chebyshev best approximation to f on E by polynomials of degree $\le \nu$ with respect to both variables. So we have to consider the measures $E_k^*(f, E)$, $k = (k_1, k_2)$ of the Chebyshev best approximation to f in $E = E^{(1)} \times E^{(2)}$

by polynomials of degree $\leq k_j$ with respect to the j-th variable, j=1,2, where E_j is bounded closed set with a positive transfinite diameter $d_j = d(E^{(j)})$ in the complex z_j plane. The main object of this paper is to extend above theorem for two complex variables. To estimate the slow and fast growth of entire functions this theorem will also be extended to (*p,q*)-scale introduced by Juneja et al. ([1], [2]). Our results can also be easily extended to *n* variables.

Let D be a complex Banach space with a norm $\|.\|$. Let $f: C^2 \to D$ be an entire function. Consider the maximum of $f: S(r,f) = \sup_{|z|=r}\{\|f(z)\|\} \forall r \in R^+$. First we have

Definition 1.1. An entire function defined on C^2 is said to be (*p,q*)-order $\rho(p,q)$ and if $(b < \rho(p,q) < \infty)$ (p,q)-type $\sigma(p,q)$ if

$$\rho(p,q) = \limsup_{r\to\infty} \frac{\log^{|p|} S(r,f)}{\log^{|q|} r}, \tag{1.2}$$

$$\sigma(p,q) = \limsup_{r\to\infty} \frac{\log^{|p-1|} S(r,f)}{(\log^{|q-1|} r)^{\rho(p,q)}}, \quad o \leq \sigma(p,q) \leq \infty, \tag{1.3}$$

where $\log^{|m|} k = \exp^{|-m|} x < \infty$ with $\log^{|o|} x = \exp^{|0|} x = x.$

Definition 1.2. An entire function *f*(*z*) defined on C^2 is of index-pair (p,q) $p \geq q \geq 1$ if $b < \rho(p,q) < \infty$ and $\rho(p-1,q-1)$ is not a finite nonzero number such that

$$\limsup_{r\to\infty} \frac{\log^{[p]} S(r,f)}{\log^{[q]} r} = \rho(p,q),$$

where,

$$b = 1 \text{ if } p = q \text{ and } b = 0 \text{ if } p > q.$$

If $\rho(p,p)$ is never greater than 1 and $\rho(p',p') = 1$ for some integer $p' \geq 1$, then the index pair of $f(z)$ is defined as (m,m) where $m = \inf\{p' : \rho(p',p') = 1\}$. If $\rho(p,q)$ is never nonzero, finite and $\rho(p'',1) = 0$ for some integer $p'' \geq 1$, then the index pair of $f(z)$ is defined as $(n,1)$ where $n = \inf\{p'' : \rho(p'',1) = 0\}$. If $f(z)$ is of index-pair (p,q) then $\rho(p,q)$ is called its (p,q)-order.

Let $P_k = P_k(C^2, D)$, $k = (k_1, k_2)$ be the set of all polynomials $p: C^2 \to D$ of degree $\leq k_j$ with respect to j-th variable, respectively, j=1,2.

Let E be a compact set in C^2 and let $f: E \to D$ be a function defined and bounded on E. Set

$$E_k^*(f,E) = \inf\{\|f-p\|_E : p \in P_k\}$$

Let $E = E^{(1)} \times E^{(2)}$, when $E^{(j)}$ $(j = 1,2)$ is a compact set in C containing infinitely many different points.

Let $\eta_j^{k_j} = (\eta_{j0},\dots,\eta_{jk_j})$, $j = 1,2$, be a system of $k_j + 1$ extremal points of E_j (see [4]).

Let

$$L^{(u_j)}(z_j) = L^{(u_j)}(z_j, E_j) = \frac{(z_j - \eta_{j0})\dots}{(\eta_{ju_j} - \eta_{j0})\dots}\Bigg|_{u_j} \frac{(z_j \eta_{jk_j})}{(\eta_{ju_j} - \eta_{jk_j})},$$

where $|u_j$ means that the factor u_j is omitted.

The polynomial

$$L_k(z) = \sum_{u_1,u_2=0}^{k_1,k_2} f(\eta_{1u_1}, \eta_{2u_2}) L^{(u_1)}(z_1) L^{(u_2)}(z_2)$$

is the Lagrange interpolation polynomial for f with nodes $\eta_1^{(k_1)} \times \eta_2^{(k_2)}$ of degree $\le k_j$ with respect to the j-th variable.

The inequality

$$E_k^*(f,E) \le \|f - L_k\|_E \left(1 + \prod_{j=1}^{2}(k_j + 1)\right) E_k^*(f,E) \tag{1.4}$$

can be proved in a similar manner as Lemma 1.1. of [3].

Now we prove

Lemma 1.1. Let $k^{(\nu)} = (k_1^{(\nu)}, k_2^{(\nu)})$, $\nu = 1,2,\dots,$ be an increasing sequence such that $\min\{k_j^{(\nu)} : j = 1,2\} \to \infty$, when $\nu \to \infty$ and $k_j^{(\nu)}$ are natural numbers.

Let $E = E^{(1)} \times E^{(2)}$, where $E^{(j)}(j = 1,2)$ is a compact set with a positive transfinite diameter $d_j = d(E^{(j)})$ in the complex z_j – plane and let $p_k \in P_k\, k$ $k = (k_1,k_2)$ be polynomials such that

$p_k(z) \equiv 0$ when $k \notin \{k^{(\nu)}\}$.

If there exist $K = (K_1,K_2) \ge 0$, $u = (u_1,u_2) > 0$, $\nu_0 \in N$ and $\lambda \ge 0$ such that

$$\|p_k\|_E \le \lambda d^{k-\gamma} \left(\frac{eKu}{k - \overline{\gamma}}\right)^{(k-\overline{\gamma})/u} \quad when \quad |k| > |k^{(\nu_0)}| := a, \tag{1.5}$$

where $d = (d_1,d_2)$, γ is a fixed natural number and $\overline{\gamma} = (\gamma,\gamma) \in R^2$, then

$$f(z) = \sum_k p_k(z), \quad z \in C^2,$$

is an entire function, and for all $\varepsilon = (\varepsilon_1,\varepsilon_2) > 0$ there exists an $r^{(0)} = (r_1^{(0)}, r_2^{(0)}) \in R^2$ such that

$$\log M(r,f) \le \sum_{j=1}^{2}(K_j+\varepsilon_j)r_j^{u_j} \quad for \quad r > r^{(0)},$$

where $M(r,f) = \sup\{\|f(z)\| : z \in E_r\}, \ r > d, \ E_{r_j}^{(j)} = \{z_j : d_j\phi_j(z_j) = r_j\}, \ r_j > d_j,$ $j = 1,2, \ E_r = E_{r_1}^{(1)} \times E_{r_2}^{(2)}, \phi_j(z_j) = \phi(z_j, E^{(j)})$ be the extremal function of the compact set $E^{(j)}, \ (j = 1,2)$.

Proof. By property of extremal function $\phi(z,E)$ [4]:

$$\|p(z)\| \le \|p\|_E \phi^{\nu}(z), \ z \in C^2.$$

applied to each variable separately we get

$$\|p_k(z)\| \le \|p_k\|_E \phi^{k_1}(z_1, E^{(1)})\phi^{k_2}(z_2, E^{(2)}) \quad for \quad z \in C^2. \tag{1.6}$$

Set $\nu(r) = (2^{u_1}eK_1u_1r_1^{u_1}, 2^{u_2}eK_2u_2r_2^{u_2})$ and take $r^{(1)} = r_1^{(1)}, \ r_2^{(1)} > (1,1)$ in such a way that $\nu(r) > k^{(0)}$ for $r > r^{(1)}$. Moreover, we assume that $\nu = 0$ and $\lambda = 1$. Using (1.6), we get

$$M(r,f) \le \sum_{|k|\le a} \frac{\|p_k\|_E}{d^k} r^k + \sum_{a<|k|\le|\nu(r)|} \left(\frac{eKu}{k}\right)^{k/u} r^k + \sum_{|k|>|\nu(r)|} \frac{1}{2^{|k|}}$$

$$\le \beta r^{\alpha} + \sum_{a<|k|\le|\nu(r)|} \left(\frac{eKu}{k}\right)^{k/u} r^k + 2^2 \quad for \quad r > r^{(1)},$$

where β does not depend on $r, \alpha = (\alpha,\alpha) \in R^2$. The maximum value of the expression $\left(\frac{eK_ju_j}{k_j}\right)^{k_j/u_j} r_j^{k_j}, \ j = (1,2)$ for r_j fixed is obtained for $k_j = u_jK_jr_j^{u_j}$ and is equal to $\exp(K_jr_j^{u_j})$, we obtain

$$M(r,f) \le \beta r^{\bar{\alpha}} + \left(\begin{matrix} |\hat{\nu(r)}| + 2 \\ 2 \end{matrix}\right) - \left(\begin{matrix} \bar{\alpha}+2 \\ 2 \end{matrix}\right) \exp\left(\sum_{j=1}^{2} k_jr_j^{u_j}\right) + 2^2$$

$$\le \beta r^{\bar{\alpha}} + \frac{2|\hat{\nu(r)}|}{2!} \exp\left(\sum_{j=1}^{2} k_jr_j^{u_j}\right) + 2^2$$

$$\le \left(\frac{\beta r^{\bar{\alpha}}}{\exp\left(\sum K_jr_j^{u_j}\right)} + \frac{2|\hat{\nu(r)}|}{2!} + \frac{2^2}{\exp\left(\sum K_jr_j^{u_j}\right)}\right) \exp\left(\sum_{j=1}^{2} k_jr_j^{u_j}\right)$$

for $r > r^{(1)}$, where $\hat{\nu(r)}$ is the smallest entire number greater than or equal to $\nu(r)$. Hence we get

$$M(r,f) \le \exp\sum_{j=1}^{2}(K_j+\varepsilon_j)r_j^{u_j} \quad \text{for} \quad r > r^{(o)} .$$

Since for any $K' > K$ we have $(K'/k)^{k/u} > \left(K/(k-\bar{\gamma}\right)^{(k-\bar{\gamma})/u}$ when k is sufficiently large, in the case of $\gamma \neq 0$ or $\lambda \neq 1$ the proof is analogous with the only difference that before the second and third component of the right hand side of inequality (1.7) there occur positive constants which have no influence of the reasoning.
Now we prove our main results.

Theorem 1.1. If the transfinite diameter $d_j = d(E^{(j)}) > 0$ $(j=1,2)$ and $\rho(p,q) = (\rho_1(p,q), \rho_2(p,q)) > (b,b)$, $\sigma(p,q) = (\sigma_1(p,q), \sigma_2(p,q)) > (o,o)$, are (*p*,*q*)-order and (*p*,*q*)-type of an entire function f respectively, then

$$\frac{\sigma(p,q)}{m} = \limsup_{\min\{k_j\}\to\infty} \frac{\log^{|p-2|} k}{\left(\log^{|q-1|} E_k^*(f,E)^{-1/k}\right)^{\rho(p,q)-A}}, \tag{1.8}$$

where

$$m = \begin{cases} \dfrac{(\rho(2,2)-1)^{(\rho(2,2)-1)}}{(\rho(2,2))^{(\rho(2,2))}} & \text{if} \quad (p,q)=(2,2), \\ \dfrac{1}{e\rho(2,1)d^{\rho(2,1)}} & \text{if} \quad (p,q)=(2,1), \\ 1 & \text{Otherwise.} \end{cases}$$

and

$$A = \begin{cases} 1 & for (p,q)=(2,2) \\ 0 & Otherwise. \end{cases}; \quad b = \begin{cases} 1 & for (p,q)=(2,2) \\ 0 & Otherwise. \end{cases}, \quad k=(k_1,k_2)$$

Proof. Let $W_k(z) = \prod_{j=1}^{2}(z_j - \eta_{j0})...(z_j - \eta_{jk_j})$, where $\{\eta_{j0}, \eta_{jk_j}\}$ is a system of $k_j + 1$ extremal points of the compact set $E^{(j)}$ $(j=1,2)$.

If r_j is sufficiently large, such that $r_j > r_j^{(o)}$, then

$$E_{r_j}^{(j)} = \{z_j : d_j \phi(z_j, E^{(j)}) = r\}$$

is a union of finite number of mutually disjoint analytic Jordan curves in the complex z_j – plane, therefore

$$f(f) - L_k(z) = \frac{1}{(2\pi_1)^2} \iint_{E_{r_1}^{(1)} E_{r_2}^{(2)}} \frac{W_k(z) f(\zeta)}{W_k(\zeta)(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta,$$

where $d\zeta = d\zeta_1 d\zeta_2$.
It can be easily seen [5] that for every $\varepsilon_j > 0$ there exist $\lambda_j, r_j^{(1)}$ and $K_j^{(1)}$ such that

$$\left|\frac{1}{2\pi i}\int_{E_{rj}^{(j)}}\left\|\frac{(z_j-\eta_{j0})\ldots(z_j-\eta_{jk_j})d\zeta_j}{(\zeta_j-\eta_{j0})\ldots(\zeta_j-\eta_{jk_j})(\zeta_i-z_j)}\right\|\right| \leq \lambda_j\left(\frac{d_j e^{\epsilon_j}}{r_j}\right)^{k_j} \quad for \quad r_j > r_j^{(1)}, \; k_j > k_j^{(1)}.$$

Using (1.9) we have

$$\|f-L_k\|_E \leq \lambda\frac{M(r,f)}{r^k}(de^E)^k, \tag{1.10}$$

for $r > r^{(1)} = (r_1^{(1)}, r_2^{(1)})$, $k > k^{(1)} = (k_1^{(1)}, k_2^{(1)})$, where $\lambda = \lambda_1, \lambda_2$, $\varepsilon = (\varepsilon_1, \varepsilon_2)$, $e^n = (e^{E_1}, e^{E_2})$.

Let $K(p,q) = (K_1(p,q), K_2(p,q)) > \sigma(p,q)$. By the definition of (*p,q*)-type of *f* and in view of ([3], eq. 1.7) there exists an $r^{(2)} > r^{(1)}$ such that

$$\frac{\log^{[p-1]} M(r,f)}{(\log^{[q-1]} r)^{\rho(p,q)}} \leq K(p,q) \quad for \quad r > r^{(2)},$$

or

$$M(r,f) \leq \exp^{[p-1]}\left[K(p,q)(\log^{[q-1]} r)^{\rho(p,q)}\right] \tag{1.11}$$

For (*p,q*)=(2,1), using (1.11) with (1.10) we get

$$\|f-L_k\|_E \leq \lambda(de^{\varepsilon})^k\left[\frac{e^{K(2,1)}r^{\rho(2,1)}}{r^k}\right]. \tag{1.12}$$

Let $k^{(2)} > k^{(1)}$ such that

$$\left(\frac{k_j}{r_j\rho_j(2,1)}\right)^{1/\rho_j(2,1)} > r_j \quad for \quad j=1,2, \; k > k^{(2)}.$$

Choosing

$$r = \left(\left(\frac{k_1}{K_2(2,1)\rho_2(2,1)}\right)^{1/\rho_1(2,1)}, \left(\frac{k_2}{K_2(2,1)\rho_2(2,1)}\right)^{1/\rho_2(2,1)}\right)$$

in (1.12), we get

$$\|f-L_k\|_E \leq \lambda((de^{\varepsilon})^k\left(\frac{eK(2,1)\rho(2,1)}{k}\right)^{k/\rho(2,1)}$$

$$\leq \lambda d^k\left(\frac{e\sigma(2,1)\rho(2,1)}{k}\right)^{k/\rho(2,1)}(e^{\varepsilon-\delta/k})^k \quad for \quad k > k^{(2)},$$

where $\delta = (\delta_1, \delta_2)$. Which gives

$$k\left(\|f-L_k\|_E\right)^{\rho(2,1)/k} \leq e\rho(2,1)\sigma(2,1)d^{\rho(2,1)}(\lambda^{\rho(2,1)/k}(e^{\varepsilon-\delta/k})^{\rho(2,1)})$$

or

$$\frac{k}{\left(\|f-L_k\|_E^{-1/k}\right)^{\rho(2,1)}} \leq \frac{\rho(2,1)}{m}. \tag{1.13}$$

For $(p,q)=(2,2)$, from (1.11) we have

$$M(r,f) \leq \exp\left[K(2,2)(\log r)^{q(2,2)}\right],$$

and by (1.10), we have

$$\|f-L_k\|_E \leq \lambda(de^{\varepsilon})^k \exp\left[K(2,2)(\log r)^{q(2,2)}\right]\left(\frac{1}{r^k}\right). \tag{1.14}$$

Let $k^{(2)} > k^{(1)}$ such that $r = \left(\frac{k_j}{K(2,2)\rho_j(2,2)}\right)^{1/(\rho_j(2,2)-1)} > r_j,\ (j=1,2),\ k > k^{(2)}$.

Choosing

$$r = \left(\exp\left(\frac{k_1}{K_2(2,2)\rho_2(2,2)}\right)^{1/(\rho_1(2,2)-1)}, \exp\left(\frac{k_2}{K_2(2,2)\rho_2(2,2)}\right)^{1/(\rho_2(2,2)-1)}\right)$$

in (1.14), we get

$$\|f-L_k\|_E \leq \frac{\lambda(de^{\varepsilon})\left\{\exp\left[\left(\frac{k}{\rho(2,2)}\right)^{\rho/(2,2)/(\rho(2,2)-1)} \frac{1}{(K(2,2))^{1/(\rho(2,2)-1)}}\right]\right\}}{\left\{\exp\left[\left(\frac{k_2}{K(2,2)\rho(2,2)}\right)^{1/(\rho(2,2)-1)}\right]\right\}^k}$$

or

$$\log\|f-L_k\|_E \leq \log\lambda + \left(\frac{k}{\rho(2,2)}\right)^{\rho/(2,2)/(\rho(2,2)-1)} \frac{1}{(K(2,2))^{1/(\rho_2(2,2)-1)}} + k\log d$$
$$+ k\varepsilon - k\left(\frac{k}{K(2,2)\rho(2,2)}\right)^{1/(\rho_2(2,2)-1)}$$

or

$$-\frac{1}{k}\log\|f-L_k\|_E \geq \left(\frac{k}{K(2,2)\rho(2,2)}\right)^{1/((\rho(2,2)-1)} - \left(\frac{1}{\rho(2,2)}\right)^{\rho(2,2)/(\rho(2,2)-1)} \left(\frac{k}{K(2,2)}\right)^{1/(\rho(2,2)-1)}$$
$$-\frac{1}{k}\log\lambda - \log d - \varepsilon$$
$$= \left(\frac{k}{K(2,2)\rho(2,2)}\right)^{1/((\rho(2,2)-1)}\left[1-\left(\frac{1}{\rho(2,2)}\right)\right] - \frac{1}{k}\log\lambda - \log(de^{\varepsilon})$$

$$= \left(\frac{k}{K(2,2)\rho(2,2)}\right)^{1/((\rho(2,2)-1)} \left(\frac{\rho(2,2)-1}{\rho(2,2)}\right)[1-O(1)]$$

or

$$\left[\log\|f-L_k\|_E^{-1/k}\right]^{\rho(2,2)-1} \geq \frac{k}{K(2,2)}\left(\frac{(\rho(2,2)-1)^{(\rho(2,2)-1)}}{(\rho(2,2))^{\rho(2,2)}}\right)[1-O(1)]^{\rho(2,2)-1}$$

or

$$\frac{K(2,2)}{m} \geq \limsup_{\min\{k_j\}\to\infty} \frac{k}{\left[\log\|f-L_k\|_E^{-1/k}\right]^{\rho(2,2)-1}} \tag{1.15}$$

Now we consider the case when $(p,q) \neq (2,1)$ and $(2,2)$ i.e., $3 \leq q \leq p < \infty$, let $k^{(2)} > k^{(1)}$ such that

$$\exp^{[q-1]}\left[\frac{\log^{[p-2]}(k_j / K(p,q)\rho_j(p,q))}{K(p,q)}\right]^{1/\rho_j(p,q)} > r_j \quad for \quad k > k^{(2)},\ j = 1,2.$$

Choosing

$$r = (\exp^{[q-1]}\left[\frac{\log^{[p-2]}(k_1 / K_1(p,q)\rho_1(p,q))}{K_1(p,q)}\right]^{1/\rho_1(p,q)},$$

$$\exp^{[q-1]}\left[\frac{\log^{[p-2]}(k_2 / K_2(p,q)\rho_2(p,q))}{K_2(p,q)}\right]^{1/\rho_2(p,q)})$$

in (1.10) and (1.11), we get

$$\|f-L_k\|_E \leq \frac{\lambda(de^{\varepsilon})^k \exp(k/K(p,q)\rho(p,q))}{\left\{\exp^{[q-1]}\left[\frac{\log^{[p-2]}(k/K(p,q)\rho_j(p,q))}{K(p,q)}\right]^{1/\rho(p,q)}\right\}^k}$$

or

$$\log\|f-L_k\|_E \leq (k/K(p,q)\rho(p,q)) + k\log d + k\varepsilon + \log\lambda - k\exp^{[q-2]} \times \left[\frac{\log^{[p-2]}(k/K(p,q)\rho(p,q))}{K(p,q)}\right]^{1/\rho(p,q)}$$

or

$$\log\|f-L_k\|_E^{-1/k} \geq \exp^{[q-2]}\left[\frac{\log^{[p-2]}(k/K(p,q)\rho(p,q))}{K(p,q)}\right]^{1/\rho(p,q)} [1-o(1)]$$

for sufficiently large values of k_j's, or

$$\left[\log^{[q-1]}\|f-L_k\|_E^{-1/k}\right]^{\rho(p,q)} \geq \left[\frac{\log^{[p-2]}(k/K(p,q)\rho_j(p,q))}{K(p,q)}\right][1-o(1)]^{\rho(p,q)},$$

or since $p > 2$,

$$K(p,q) \geq \frac{\log^{[p-2]} k}{\left[\log^{[q-1]} \|f - L_k\|_E^{-1/k}\right]^{\rho(p,q)}} [1 - o(1)]^{\rho(p,q)}.$$

Proceeding to limits, we get

$$\limsup_{\min\{k_j\} \to \infty} \frac{\log^{[p-2]} k}{\left[\log^{[q-1]} \|f - L_k\|_E^{-1/k}\right]^{\rho(p,q)}} \leq K(p,q). \tag{1.16}$$

Since (1.15) and (1.16) are valid for every $K(p,q) = (K_1(p,q), K_2(p,q)) > \sigma(p,q)$, follows that

$$\limsup_{\min\{k_j\} \to \infty} \frac{\log^{[p-2]} k}{\left[\log^{[q-1]} \|f - L_k\|_E^{-1/k}\right]^{\rho(p,q)-A}} \leq \frac{\sigma(p,q)}{m}. \tag{1.17}$$

To prove reverse inequality, let $\tilde{v} = (v,v) \in R^2$, $v = 0,1,\ldots$ and in view of Lemma 1.1 expanding to the function f in the series

$$f(z) = L_{\tilde{o}}(z) + \sum_{v=0}^{\infty} \left(L_{\tilde{v}+1}(z) - L_{\tilde{v}}(z)\right), \quad z \in C^2$$

we obtain

$$\begin{aligned} \|L_{\tilde{v}+1} - L_{\tilde{v}}\| &\leq \|f - L_{\tilde{v}+1}\|_E + \|f - L_{\tilde{v}}\|_E \\ &\leq 2\|f - L_{\tilde{v}}\|_E \end{aligned}$$

or

$$\|f(z)\| \leq \|L_{\tilde{o}}(z)\| + \sum_{v=0}^{\infty} \|L_{\tilde{v}+1}(z) - L_{\tilde{v}}(z)\|,$$

using property (1.6) of extremal function and applied to every variable seperately, we have

$$\|f(z)\| \leq a_0 + 2\sum_{v=0}^{\infty} \|f - L_{\tilde{v}}\|_E (r/d)^{\tilde{v}} \quad for \quad z \in E_{r_j}^{(j)}. \tag{1.18}$$

Consider the function

$$g(z) = \sum_{v=0}^{\infty} \|f - L_{\tilde{v}}\|_E z^{\tilde{v}}.$$

Since $\lim_{\tilde{v} \to \infty} \|f - L_{\tilde{v}}\|_E^{1/\tilde{v}} = 0$ in view of Lemma 1.1, it follows that $g(z)$ is entire function and further (1.18) gives

$$M(r,f) \leq a_0 + 2g(r/d). \tag{1.19}$$

which gives

$$\frac{\sigma(p,q)}{m} \leq \limsup_{\min\{\tilde{v}_j\} \to \infty} \frac{\log^{[p-2]} \tilde{v}}{\left[\log^{[q-1]} \|f - L_{\tilde{v}}\|_E^{-1/\tilde{v}}\right]^{\rho(p,q)-A}}. \tag{1.20}$$

Using inequality (1.4) with (1.17) and (1.20) together gives the proof of theorem.

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