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SPECTRAL DECOMPOSITIONS AND FEASIBLE DIRECTIONS IN THE AXIAL THREE-INDEX ASSIGMENT PROBLEM

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ABSTRACT

In this paper, we give some results on the spectral decomposition and generalized inverse of the matrix A which is the coefficient matrix of the axial three-index assignment problem and investigate relations between eigenvalues and eigenvectors of the matrices AA^T and $I-A^+A$ where A^T is the transpoze and A^+ is the generalized inverse of A. It has been shown that the feasible directions of the axial three-index assignment problem can be investigated in terms of the eigenvectors of the matrix AA^T.

1. INTRODUCTION

Axial three-index assignment problem (AP) is a special case of general linear programming problem. AP is a close relative of the (axial) 3-dimensional transportation problem (TP). TP were first studied by Schell in [11] For further references concerning these problems see [1,4,6,8]. Application of AP was mentioned in [8,9] and also among the early algorithms and heuristics or this problem are those of [8,9].

In this study, we investigate algebraic characterizations of the singular value decompositions in the AP using the paper which was applied to the transportation problem by Bulut in 1991 [4]. It is also shown that the feasible directions of AP can be investigated in terms of the eigenvectors of the matrix AA^T and is obtained further results among the eigenvectors of this matrix.

AP can be stated as

$$\text{Minimize} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk}x_{ijk}$$

$$\text{subject to } \sum_{j=1}^{n}\sum_{k=1}^{n} x_{ijk} = 1, \; i = 1,\dots,n$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n} x_{ijk} = 1,\ j = 1,\dots,n$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ijk} = 1,\ k = 1,\dots,n$$

where $x_{ijk} = 1$ or 0 for all i,j,k=1,...,n.

This problem can be formulated in the matrix form

$$\mathrm{Min}\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{1}_{3n}^T,\ \mathbf{x} \geq 0\}$$

where

$$A = \begin{bmatrix} 1_n \otimes 1_n \otimes I_n \\ 1_n \otimes I_n \otimes 1_n \\ I_n \otimes 1_n \otimes 1_n \end{bmatrix}$$

$\mathbf{x}^T = [x_{111}, x_{112}, \dots, x_{nnn}]$, $\mathbf{c}^T = [c_{111}, c_{112}, \dots, c_{nnn}]$ and I_n is $n \times n$ identity matrix, 1_n is the $1 \times n$ vector whose all entries are 1 and $\otimes$ is the Kronecker product.

2. BASIC DEFINITIONS AND THEOREMS

In this section, we give the basic definitions and theorems without proof.

Definition 2.1. (The Moree-Penrose inverse) The generalized inverse [3,4,5,6] A^+ of an arbitrary $m \times n$ matrix A, is uniquely determined as that matrix which simultaneously satisfies the following system of four matrix equations:

$$AA^+A = A,\ A^+AA^+ = A^+,\ (AA^+)^T = AA^+,\ (A^+A)^T = A^+A$$

Theorem 2.1. [3,4,5] Any $m \times n$ matrix A, of rank r, can be written as

$$A = \sum_{i=1}^{r} \lambda_i E_i \tag{2.1}$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$ are the singular values of A, i.e., positive square roots of the positive eigenvalues of A^TA and AA^T and the matrices

$$E_i = \mathbf{u}_i\mathbf{v}_i,\ i = 1,2,\dots,r \tag{2.2}$$

satisfy

$$E_iE_j^T = 0,\ E_i^TE_j = 0,\ 1 \leq i \neq j \leq r \tag{2.3}$$

$$E_iE_i^TE_i = E_i,\ \forall i$$

and where

$$AA^T\mathbf{u}_i = \lambda_i\mathbf{u}_i,\ \forall i \tag{2.4}$$

$$A^T\mathbf{u}_i = \lambda_i\mathbf{v}_i,\ \forall i \tag{2.5}$$

The matrices E_i in Theorem 2.1 are partial isometries [3,4,7] and Eq.(2.1) is the singular value decomposition of the matrix A. Furthermore

$$A^{+} = \sum_{i=1}^{r} \frac{1}{\lambda_i} E_i^{T} \tag{2.6}$$

$$AA^{+} = \sum_{i=1}^{r} u_i u_i^{T} \quad \text{and} \quad A^{+}A = \sum_{i=1}^{r} v_i v_i^{T} \tag{2.7}$$

Theorem 2.2.[4,7] Consider the partitioned matrix A=[B,C] and let $N=(I-BB^{+})C$ and $K=(I+C^{T}(B^{+})^{T}B^{+}C)^{-1}$ then

$$A^{+} = \begin{bmatrix} B^{+} - B^{+}CKC^{T}(B^{+})^{T}B^{+} \\ N^{+} + KC^{T}(B^{+})^{T}B^{+} \end{bmatrix}$$

if and only if $N^{+}NC^{T}(B^{+})^{T}B^{+}C = 0$.

Definition 2.2. [2.10] Consider the linear programming problem

$$\text{Min}\left\{c^{T}\mathbf{x} \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0\right\}$$

where A is $m \times n$ matrix, **b** is an $m \times 1$ vector. Let x_0 be a basic feasible solution. Then a nonzero vector **d** is a feasible direction at x_0 if and only if

$$A\mathbf{d}=0 \text{ and } d_j \geq 0, \text{ if } x_{0j} = 0$$

where x_{0j} is the jth component of x_0 and d_j is the j-th component of **d**.

Corollary 2.3. [2,10] For the linear programming problem $\text{Min}\left\{c^{T}\mathbf{x} \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0\right\}$ the feasible direction at a feasible point $\mathbf{x}_0$ can be investigated in terms of the eigenvectors corresponding to zero eigenvalues of the matrix $I-A^{+}A$.

3. SINGULAR VALUE DECOMPOSITION OF THE MATRIX A AND SOME RESULTS

In this section we obtain some results about spectral decomposition of the $3n \times n^3$ matrix A, of rank 3n-2, given in (1.2).
Using the matrices A and A^T, we compute the matrices AA^T and A^TA as follows

$$AA^{T} = \begin{bmatrix} n^2 I_n & nJ_n & nJ_n \\ nJ_n & n^2 I_n & nJ_n \\ nJ_n & nJ_n & n^2 I_n \end{bmatrix} \tag{3.1}$$

$$A^{T}A = J_n \otimes J_n \otimes I_n + J_n \otimes I_n \otimes J_n + I_n \otimes J_n \otimes J_n \tag{3.2}$$

where J_n is $n \times n$ matrix whose all entries are 1.

The characteristic equations of the matrices AA^T and A^TA are obtained as follows

$$\det(AA^{T} - \lambda I) = \lambda^2 (n^2 - \lambda)^{3n-3}(3n^2 - \lambda) = 0 \tag{3.3}$$

$$\det(A^TA - \lambda I) = \lambda^{n^3-3n+2}(n^2 - \lambda)^{3n-3}(3n^2 - \lambda) = 0 \quad (3.4)$$

Let $\{\mathbf{u}_1,...,\mathbf{u}_{3n-2}\}$ and $\{v_1,...,v_{3n-2}\}$ the sets of eigenvectors corresponding to nonzero eigenvalues of the matrices AA^T and A^TA, respectively.

Corollary 3.1. The unit eigenvector corresponding to eigenvalue $3n^2$ of AA^T is

$$u_1^T = \frac{1}{\sqrt{3n}}[1_n\ 1_n\ 1_1]$$

Proof. Let $\mathbf{u}_1^T = [\mathbf{x}_1^T, \mathbf{y}_1^T, \mathbf{z}_1^T]$. Using $AA^T\mathbf{u}_1 = 3n^2\mathbf{u}_1$, we obtain $J_n x_1 = n x_1$, $J_n y_1 = n y_1$, $J_n z_1 = n z_1$. Hence the proof is easily completed.

Corollary 3.2. The orthonormal eigenvectors corresponding to eigenvalues n^2 of AA^T are

$$\mathbf{u}_i^T = [\mathbf{x}_i^T, 0^T, 0^T], \text{ for } i = 2,...,n$$

$$\mathbf{u}_i^T = [0^T, \mathbf{y}_i^T, 0^T], \text{ for } i = n+1,...,2n-1$$

$$\mathbf{u}_i^T = [0^T, 0^T, \mathbf{z}_i^T], \text{ for } i = 2n,...,3n-2$$

where $\mathbf{x}_i$, $\mathbf{y}_i$ and $\mathbf{z}_i$, $\forall i$, are the linearly independent solutions of the $J_n\mathbf{x}=0$ and $J_n\mathbf{y}=0$ and $J_n\mathbf{z}=0$ respectively.

Proof. Let $\mathbf{u}^T = [\mathbf{x}^T, \mathbf{y}^T, \mathbf{z}^T]$. From the equality $AA^T\mathbf{u} = n^2\mathbf{u}$, we have $J_n\mathbf{x} = 0$, $J_n\mathbf{y} = 0$ and $J_n\mathbf{z} = 0$, and the corollary is trivial.

Corollary 3.3. The eigenvectors corresponding to eigenvalues $3n^2$ and n^2 of the matrix A^TA defined in (3.2) are as follows, respectively

$$v_1 = \frac{1}{n^{3/2}}\mathbf{1}_n^T \otimes \mathbf{1}_n^T \otimes \mathbf{1}_n^T$$

and

$$v_i = \frac{1}{n}(\mathbf{1}_n^T \otimes \mathbf{1}_n^T \otimes I_n^T)\mathbf{x}_i, \text{ for } i=2,...,n$$

$$v_i = \frac{1}{n}(\mathbf{1}_n^T \otimes I_n^T \otimes \mathbf{1}_n^T)\mathbf{y}_i \text{ for } i=n+1,...,2n-1$$

$$v_i = \frac{1}{n}(I_n^T \otimes \mathbf{1}_n^T \otimes \mathbf{1}_n^T)\mathbf{z}_i \text{ for } i=2n,...,3n-2$$

where $\mathbf{x}_i$, $\mathbf{y}_i$ and $\mathbf{z}_i$ are the linearly independent solutions of the eqs. $J_n\mathbf{x} = 0$, $J_n\mathbf{y} = 0$ and $J_n\mathbf{z} = 0$.

Proof. Using the Eq.(2.5) Corollary 3.1 and Corollary 3.2 it can be easily proved.

Corollary 3.4. Let $\mathbf{u}_i$ and $\mathbf{v}_i$ for i=1,...,3n-2, be eigenvectors of AA^T and A^TA respectively. Then

$$A = \begin{bmatrix} 1_n \otimes 1_n \otimes (\frac{1}{n}J_n + XX^T) \\ 1_n \otimes (\frac{1}{n}J_n + YY^T) \otimes 1_n \\ (\frac{1}{n}J_n + ZZ^T) \otimes 1_n \otimes 1_n \end{bmatrix} \tag{3.5}$$

where $X = [x_2, x_3, ..., x_n]$, $Y = [y_2, y_3, ..., y_n]$ and $Z = [z_2, z_3, ..., z_n]$

Proof. Using the Corollaries 3.1, 3.2 and 3.3 and Eq. (2.2) we get

$$E_1 = \frac{1}{\sqrt{3}\,n^2} \begin{bmatrix} 1_n \otimes 1_n \otimes J_n \\ 1_n \otimes J_n \otimes 1_n \\ J_n \otimes 1_n \otimes 1_n \end{bmatrix}$$

and

$$E_i = \frac{1}{n} \begin{bmatrix} x_i x_i^T (1_n \otimes 1_n \otimes I_n) \\ 0 \\ 0 \end{bmatrix}, \quad \text{for } i = 2, ..., n$$

$$E_i = \frac{1}{n} \begin{bmatrix} 0 \\ y_i y_i^T (1_n \otimes I_n \otimes 1_n) \\ 0 \end{bmatrix}, \quad \text{for } i = n+1, ..., 2n-1$$

$$E_i = \frac{1}{n} \begin{bmatrix} 0 \\ 0 \\ z_i z_i^T (I_n \otimes 1_n \otimes 1_n) \end{bmatrix}, \quad \text{for } i = 2n, ..., 3n-2$$

and hence the matrix A can be easily obtained using the Eq. (2.1).

If we compare the matrices given in (1.2) and (3.5) we have

$$XX^T = YY^T = ZZ^T = I_n - \frac{1}{n}J_n.$$

4. THE CHARACTERIZATION OF THE FEASIBLE DIRECTIONS

Using Theorem 2.2, the Moore-Penrose inverse A^+ of A given in (1.2) is obtained in the form

$$A^{+} = \frac{1}{3n^3}\left[1_n{}^T \otimes 1_n{}^T \otimes (3nI_n - 2J_n)\, 1_n{}^T \otimes (3nI_n - 2J_n) \otimes 1_n{}^T (3nI_n - 2J_n) \otimes 1_n{}^T\right] \quad (4.1)$$

Thus we have

$$A^{+}A = \frac{1}{n^2}(J_n \otimes J_n \otimes I_n + J_n \otimes I_n \otimes J_n + I_n \otimes J_n \otimes I_n) - \frac{2}{n^3} J_n \otimes J_n \otimes J_n \quad (4.2)$$

$$I - A^{+}A = I_n \otimes I_n \otimes I_n - \frac{1}{n^2}(J_n \otimes J_n \otimes I_n + J_n \otimes I_n \otimes J_n + I_n \otimes J_n \otimes I_n) + \frac{2}{n^3} J_n \otimes J_n \otimes J_n \quad (4.3)$$

and

$$\det(A^{+}A - \lambda I) = \lambda^{n^3 - 3n + 2}(\lambda - 1)^{3n-2} \quad (4.4)$$

Corollary 4.1. The eigenvectors corresponding to eigenvalues 1 of the matrix $I - A^{+}A$ are in the form

$$\begin{aligned} \mathbf{u}_i &= \mathbf{x}_j \otimes \mathbf{y}_k \otimes \mathbf{z}_t, && \text{for } i = 1,2,\dots,(n-1)^3 \\ \mathbf{u}_i &= \mathbf{x} \otimes \mathbf{y}_k \otimes \mathbf{z}_t, && \text{for } i = (n-1)^3 + 1,\dots,(n-1)^3 + (n-1)^2 \\ \mathbf{u}_i &= \mathbf{x}_j \otimes \mathbf{y} \otimes \mathbf{z}_t, && \text{for } i = (n-1)^3 + (n-1)^2 + 1,\dots,(n-1)^3 + 2(n-1)^2 \\ \mathbf{u}_i &= \mathbf{x}_j \otimes \mathbf{y}_k \otimes \mathbf{z}, && \text{for } i = (n-1)^3 + 2(n-1)^2 + 1,\dots,n^3 - 3n + 2 \end{aligned} \quad (4.5)$$

where $\mathbf{x}_j$, $\mathbf{y}_k$ and $\mathbf{z}_t$, $2 \le j, k,t \le n$, are the eigenvectors corresponding to eigenvalues 0 of the matrix J_n and **x, y** and **z** are the eigenvectors corresponding to eigenvalue n of J_n .

Proof. Since $\text{rank}(A^{+}A) = \text{rank}(A) = 3n - 2$ and $A^{+}A$ has size $n^3 \times n^3$ the Eq. $A^{+}A=0$ has n^3-3n+2 linearly independent solutions. It can be shown that the linearly independent eigenvectors as in (4.5) satisfy the eq. $A^{+}A\mathbf{u}=0$.

Corollary 4.2. The feasible directions of the problem given in (1.2) can be investigated in terms of the eigenvectors of the matrix J_n.

Proof. Using Theorem 2.3 and Corollary 4.1, it is easily seen.

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