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SOME NEW GENERALIZED SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

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ABSTRACT

In this paper, we introduce some new generalized sequence spaces using Orlicz function. We also examine some properties of these sequence space.

1. INTRODUCTION

Let l_∞, c and c_o be the Banach spaces of bounded, convergent and null sequences $x=(x_k)$, respectively, normed, as usual, by $\|x\| = \sup_k |x_k| < \infty$.

Lindenstrauss and Tzafriri [1] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x : \sum_k M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad \text{for some } \rho > 0 \right\}.$$

The space l_M with the norm

$$\|x\| = \text{Inf}\left\{ \rho > 0 : \sum_k M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space.

The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x)=x^p, 1\le p<\infty$.

In the present note we introduce and examine some properties of four sequence spaces defined by using Orlicz function M, which generalize the well known Orlicz sequence space l_M and $l_\infty(p,s)$, c(p,s) and $c_o(p,s)$.

An Orlicz function is a function $M:[0,\infty[\to[0,\infty[$, which is continuous, non-decreasing and convex with M(0)=0, M(x) >0 for x>0 and $M(x)\to\infty$ as $x\to\infty$.

2. MAIN RESULTS

Let $p=(p_k)$ be a sequence of positive real numbers. We define the following sequence spaces

$$l_M(p,s)=\left\{x\in w:\ \sup_n \sum_k k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k}<\infty,\quad \text{for some } \rho>0 \text{ and } s\ge 0\right\},$$

$$l_\infty(M,p,s)=\left\{\begin{array}{l} x\in w:\sup_{n,k} k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k}<\infty, \\ \text{for some } \rho>0 \text{ and } s\ge 0\end{array}\right\},$$

$$c(M,p,s)=\left\{\begin{array}{l} x\in w:k^{-s}\left[M\left(\frac{|x_{k+n}-L|}{\rho}\right)\right]^{p_k}\to 0 \text{ as } k\to\infty, \\ \text{for some} \rho, L>0, \text{ and } s\ge 0, \text{ uniformly in } n\end{array}\right\},$$

$$c_o(M,p,s)=\left\{\begin{array}{l} x\in w:k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k}\to 0 \text{ as } k\to\infty, \\ \text{for some } \rho>0, \text{ and } s\ge 0, \text{ uniformly in } n\end{array}\right\}.$$

When $p_k=1$ for all k, n=0 and s=0, then $l_M(p,s)$ becomes l_M. When M(x)=x and s=0 then the family of sequences defined above become $l(p)$, $l_\infty(p)$, $c(p)$ and $c_o(p)$ respectively [2].

When M(x)=x and n=0, then $l_M(p,s)$ becomes $l(p,s)$ which has been investigated by Bulut and Çakar [3] and $l_\infty(M,p,s)$, $c(M,p,s)$ and $c_0(M,p,s)$ become $l_\infty(p,s)$, $c(p,s)$ and $c_0(p,s)$ which has been investigated by Başarir [4].

In order to discuss the properties of $l_M(p,s)$, we assume that $p=(p_k)$ is bounded.

Theorem 1. Let $H=\sup_k p_k<\infty$ then $l_M(p,s)$ is a linear set over the set of complex numbers **C**.

Proof. Let $x,y\in l_M(p,s)$ and $\alpha,\beta\in \mathbf{C}$. In order to prove the result we need to find some ρ_3 and $s\geq 0$ such that

$$\sup_n \sum_k k^{-s}\left[M\left(\frac{|\alpha x_{k+n}+\beta y_{k+n}|}{\rho_3}\right)\right]^{p_k}<\infty .$$

Since $x,y\in l_M(p,s)$, therefore there exists some positive ρ_1 , ρ_2 and $s\geq 0$ such that

$$\sup_n \sum_k k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho_1}\right)\right]^{p_k}<\infty,$$

and

$$\sup_n \sum_k k^{-s}\left[M\left(\frac{|y_{k+n}|}{\rho_2}\right)\right]^{p_k}<\infty .$$

Define $\rho=\max(2|\alpha|\rho_1,2|\beta|\rho_2)$. Since M is non-decreasing and convex

$$\sum_k k^{-s}\left[M\left(\frac{|\alpha x_{k+n}+\beta y_{k+n}|}{\rho_3}\right)\right]^{p_k}\leq\sum_k k^{-s}\left[M\left(\frac{|\alpha x_{k+n}|}{\rho_3}+\frac{|\beta y_{k+n}|}{\rho_3}\right)\right]^{p_k}$$

$$\leq\sum_k k^{-s}\frac{1}{2^{p_k}}\left[M\left(\frac{|x_{k+n}|}{\rho_1}+\frac{|y_{k+n}|}{\rho_2}\right)\right]^{p_k}$$

$$< \sum_k k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho_1}\right)+M\left(\frac{|y_{k+n}|}{\rho_2}\right)\right]^{p_k}$$

$$\leq C.\sum_k k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho_1}\right)\right]^{p_k}+C.\sum_k k^{-s}\left[M\left(\frac{|y_{k+n}|}{\rho_2}\right)\right]^{p_k} \quad \text{for all n,}$$

where C=max $(1,2^{H-1})$. This proves that $l_M(p,s)$ is linear.

Theorem 2. $l_M(p,s)$ is paranormed space with the paranorm

$$G(x)=\text{Inf}\left\{\rho^{p_n/H} : \left(\sum_k k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k}\right)^{1/H} \leq 1,\ n=1,2,\ldots,s\geq 0\right\}$$

where H=max$(1,\sup_k p_k)$.

Proof. Clearly G(x)=G(-x). The subadditivitity of G follows from Theorem 1. Since M(0)=0, we get $\text{Inf}\left\{\rho^{p_n/H}\right\}$=0 for x=0. Conversely, suppose that G(x)=0. Then it is easy to see that x=0. Finally using the same technique of Theorem 2 of Parashar and Choudhary [5], it can be easily seen that scalar multiplication is continuous. This completes the proof.

Remark. It can easily be verified that when M(x)=x and n=0, the paranorm defined $l_M(p,s)$ and paranorm defined in $l\ (p,s)$ are the same.

An Orlicz function M can always be represented by Krasnoselskii and Rutitsky [6] in the following integral form

$$M(x) = \int_0^x q(t)dt$$

where q, known as the kernel of M, is right-differentiable for $t \geq 0$, q(0)=0, q(t)>0 for t>0, q is non-decreasing and $q(t) \to \infty$ as $t \to \infty$.

Theorem 3. Let $1 \leq p_k < \infty$. Then $l_M(p,s)$ is complete paranormed space with

$$G(x)=\operatorname{Inf}\left\{\rho^{p_n/H} : \left(\sum_k k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k}\right)^{1/H} \leq 1, n = 1,2,\ldots, s \geq 0\right\}.$$

Proof. The proof follows on the same lines as adopted by Parashar and Choudhary [5, Theorem 3]. So we omit it.

Theorem 4 (i). Let $0 < p_k \leq q_k < \infty$ for each k. Then $l_M(p,s) \subset l_M(q,s)$.

(ii). $s_1 \leq s_2$ implies $l_M(p,s_1) \subset l_M(p,s_2)$.

Proof (i). Let $x \in l_M(p,s)$. Then there exists some $\rho > 0$ and $s \geq 0$ such that

$$\sup_n \sum_k k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k} < \infty .$$

This implies that

$$i^{-s}\left[M\left(\frac{|x_{i+n}|}{\rho}\right)\right] \leq 1 \text{ for sufficiently large values of i and all n.}$$

Since M is non-decreasing, we get

$$\sum_k k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{q_k} \leq \sum_k k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k} < \infty.$$

Thus we get $x \in l_M(q,s)$.

(ii). Let $s_1 \leq s_2$ .Then $k^{-s_2} \leq k^{-s_1}$ for all k. Since

$$k^{-s_2}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k} \leq k^{-s_1}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k} \quad \text{for all n,}$$

and then

$$\sum_k k^{-s_2}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k} \leq \sum_k k^{-s_1}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k}$$

this inequality implies that $l_M(p,s_1) \subset l_M(p,s_2)$.

Definition. (Krasnoselskii and Rutitsky [6]) An Orlicz function is said to satisfy Δ_2-condition for all values of u, if there exists a constant K>0, such that

$$M(2u) \leq KM(u),\ u \geq 0.$$

The Δ_2-condition is equivalent to the satisfaction of the inequality

$$M(Lu) \leq KLM(u)$$

for all values of u and for L >1.

Theorem 5. Let M be an Orlicz function which satisfies Δ_2-condition. Then

(i). $l_\infty \subset l_M(p,s)$,

(ii). $l(p,s) \subset l_M(p,s)$.

Proof (i). Let $x \in l_\infty$. This implies that $|x_{k+n}| \leq N$ for all k and n. So that

$$k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k} \leq k^{-s}\left[M\left(\frac{N}{\rho}\right)\right]^{p_k} \leq k^{-s}[KLM(N)]^{H} \text{ by } \Delta_2\text{-}$$

condition, where $H=\max(1,\sup_k p_k)$. Hence

$$\sum_k k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k} < \infty \quad .$$

This shows that $l_\infty \subset l_M(p,s)$.

(ii). Using the same technique of Theorem 3 by Esi [8] it is easy to prove the Theorem.

Now we investigate some properties of spaces $c_o(M,p,s)$, $c(M,p,s)$ and $l_\infty(M,p,s)$ defined earlier. We first state simple property of these spaces.

Theorem 6. Let $p=(p_k)$ be bounded. Then $c_o(M,p,s)$, $c(M,p,s)$ and $l_\infty(M,p,s)$ are linear spaces.

Proof. Omitted.

Theorem 7. Let $\sup_k p_k=H<\infty$. Then $c_o(M,p,s)$ is a linear topological space paranormed by

$$g(x)=\operatorname{Inf}\left\{\rho^{p_n/H} : \left[k^{-s}\left(M\left(\frac{|x_{k+n}|}{\rho}\right)\right)^{p_k}\right]^{1/H} \leq 1, \quad n=1,2,\ldots, s\geq 0\right\}.$$

Proof. Omitted.

Theorem 8 **(i).** Let $0<\operatorname{Inf}p_k \leq p_k \leq 1$. Then $c_o(M,p,s) \subset c_o(M,s)$, $c(M,p,s) \subset c(M,s)$ and $l_\infty(M,p,s) \subset l_\infty(M,s)$.

(ii). Let $1 \leq p_k \leq \sup_k p_k < \infty$. Then $c_o(M,s) \subset c_o(M,p,s)$, $c(M,s) \subset c(M,p,s)$, and $l_\infty(M,s) \subset l_\infty(M,p,s)$.

Proof (i). Let $x \in c_o(M,p,s)$. Since $0<\operatorname{Inf}p_k \leq 1$, we get

$$k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right] \leq k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k} \quad \text{for all n,}$$

and hence $x \in c_o(M,s)$.

(ii). Let $1 \leq p_k \leq \sup p_k < \infty$ for each k and $x \in c_o(M,s)$. Then for each $0 < \varepsilon < 1$ and for all n, there exists a positive integer N such that

$$k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right] \leq \varepsilon < 1$$

for all $k \geq N$ and $s \geq 0$. This implies that

$$k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k} \leq k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right].$$

Thus we get $x \in c_o(M,p,s)$.

The other inclusions can be treated similarly.

Theorem 9 **(i).** Let $0 < p_k \leq q_k < \infty$ and $\left(\frac{q_k}{p_k}\right)$ be bounded. Then $c_o(M,q,s) \subset c_o(M,p,s)$ and $c(M,q,s) \subset c(M,p,s)$

(ii). Let M be an Orlicz function which satisfies Δ_2-condition. Then $c_o(p,s) \subset c_o(M,p,s)$, $c(p,s) \subset c(M,p,s)$ and $l_\infty(p,s) \subset l_\infty(M,p,s)$.

(iii). $c(M_1,p,s) \cap c(M_2,p,s) \subset c(M_1+M_2,p,s)$ where M_1 and M_2 are two Orlicz functions and $s \geq 0$.

(iv). $s_1 \leq s_2$ implies $c(M,p,s_1) \subset c(M,p,s_2)$.

Proof (i). If we take

$$t_k = k^{-s}\left[M\left(\frac{|x_{k+n}|}{\rho}\right)\right]^{p_k}$$

for all n, k and $s \geq 0$, then using the same technique of Theorem 2 of Nanda [7], it is easy to prove (i).

(ii) Proof is similar to Theorem 5 (ii).

Using the same technique of Theorem 3 by Esi [9], it is easy to prove Theorem 9(iii) and (iv)

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