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APPROXIMATION BY MODIFIED BERNSTEIN-BALAZS TYPE RATIONAL FUNCTIONS

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ABSTRACT

This paper is devoted to the problem of approximation of continuously differentiable functions by the modified Balaszs-Bernstein type rational functions.

The theorem on the order of approximation, in terms of modulus of continuity, and the Voronowskaja type theorem are established.

INTRODUCTION

C. Balazs [1] introduced the Bernstein type rational functions

$$R_n(f,x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)P_{k,n}(x),\ P_{k,n}(x)=\binom{n}{k}(a_nx)^k \tag{1}$$

and proved that if f is continuous in $[0,\infty)$, $f(x)=O(e^{\alpha x})\ (x\to\infty,\alpha\in R)$ then in any interval $[0,A],(A>0)$ the estimate

$$|f(x)-R_n(f,x)|\le c_0\left[w_{2A}\left(\frac{1}{n^{1/3}}\right)+\frac{1}{n^{2/3}}\right],0\le x\le A$$

holds for sufficiently large n's with $a_n=b_n/n, b_n=n^{2/3}$, where c_0 depends only on α and A, and $w_{2A}(\cdot)$ is the modulus of continuity of f on the interval [0,2A] (see also,[2],[3]).

The following theorems on the approximation of a function by the Bernstein polynomials $B_n(f,x)$ are well known.

Theorem A (G.G. Lorentz [4], p.20). If f(x) is continuous and $w(\delta)$ is the modulus of continuity of f, then

$$|f(x)-B_n(f,x)|\le\frac{5}{4}w(n^{-1/2}).$$

Theorem B (G.G. Lorentz [4], p.21). If $w_1(\delta)$ is the modulus of continuity of $f'(x)$,

$$|f(x)-B_n(f,x)| \le \frac{3}{4}n^{-1/2}w_1(n^{-1/2}),$$

where f' is the continuous derivative of f.

In the present paper we define modified Bernstein- Balazs type rational functions and prove theorems on approximation by them.

Let f be a real, single valued function defined in $[0,\infty)$. We consider functions which have continuous derivative f' in the interval [0,A] and we modify the functions in (1) as follows

$$R_n^*(f,x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)P_{k,n}(x)+\frac{a_nx^2}{1+a_nx}f'(x), \qquad (2)$$

where a_n and b_n are suitable chosen real numbers, independent of x. We call the functions in (2) modified Bernstein-Balazs type functions.

In this paper we give an estimation for the rate of convergence of the functions in (2) and prove an asymptotic approximation theorem and show that the derivatives of the functions in (2) also converge to the derivative of the function.

In [1], C. Balazs proved the following auxiliary result:

Lemma C ([1], p. 124). If $x \ge 0$, then the following identities hold:

$$\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} P_{k,n}(x)=1 \quad (n=1,2,\ldots),$$

$$\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} (k-b_nx)P_{k,n}(x)=\frac{-a_nb_nx^2}{1+a_nx},$$

$$\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} (k-b_nx)^2P_{k,n}(x)=\frac{a_n^2b_n^2x^4+b_nx}{(1+a_nx)^2}.$$

Now, we consider the functions which have continuous derivative f' in [0,A]. Let $w_1(f',x)$ be the modulus of continuity of f' in [0,2A].

(In what follows c_i, i=1,2,... will denote constants independent of n.)

Theorem 1. Let $R_n^*(f,x)$ be the functions defined by (2) with $a_n=b_n/n, b_n=n^{2/3}$. Then the inequality

$$\left|R_n^*(f,x)-f(x)\right| \le c_2 n^{-1/3}w_1(f',n^{-1/2})$$

holds for sufficiently large n's.

Proof. By Lagrange's theorem we can write

$$f\left(\frac{k}{b_n}\right)-f(x)=\left(\frac{k}{b_n}-x\right)f'(\xi), x<\xi<\frac{k}{b_n}.$$

Considering the properties of modulus of continuity, we get

$$|f'(x)-f'(\xi)|\le w_1(f',\delta)(\delta^{-1}|x-\xi|+1), x<\xi<k/b_n,$$

and

$$|f'(x)-f'(\xi)|\le w_1(f',\delta)\left(\delta^{-1}\left|\frac{k}{b_n}-x\right|+1\right). \tag{3}$$

Evidently, we have:

$$R_n^*(f,x)-f(x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\left[f\left(\frac{k}{b_n}\right)-f(x)\right]P_{k,n}(x)+\frac{a_nx^2}{1+a_nx}f'(x)$$

$$=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\left(\frac{k}{b_n}-x\right)f'(x)P_{k,n}(x)+\frac{a_nx^2}{1+a_nx}f'(x)$$

$$+\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\left(\frac{k}{b_n}-x\right)(f'(\xi)-f'(x))P_{k,n}(x)$$

$$=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\left(\frac{k}{b_n}-x\right)(f'(\xi)-f'(x))P_{k,n}(x).$$

By (3) we can write

$$\left|R_n^*(f,x)-f(x)\right|\le\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\left|\frac{k}{b_n}-x\right|\left|f'(\xi)-f'(x)\right|P_{k,n}(x)$$

$$\le\frac{w_1(f',x)}{(1+a_nx)^n}\sum_{k=0}^{n}\left|\frac{k}{b_n}-x\right|\left(\delta^{-1}\left|\frac{k}{b_n}-x\right|+1\right)P_{k,n}(x)$$

$$=\frac{w_1(f',x)}{\delta}\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\left(\frac{k}{b_n}-x\right)^2P_{k,n}(x)$$

$$+w_1(f',x)\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\left|\frac{k}{b_n}-x\right|P_{k,n}(x).$$

Using Lemma C, we get

$$\left|R_n^*(f,x)-f(x)\right|\le\frac{w_1(f',x)}{\delta}\frac{a_n^2x^4+\frac{x}{b_n}}{(1+a_nx)^2}+w_1(f',x)S \tag{4}$$

where

$$S=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\left|\frac{k}{b_n}-x\right|P_{k,n}(x)$$

$$= \frac{1}{(1+a_n x)^n}\left(\sum_{\frac{k}{b_n}\le 2A} \left|\frac{k}{b_n}-x\right| P_{k,n}(x) + \sum_{\frac{k}{b_n}\ge 2A} \left|\frac{k}{b_n}-x\right| P_{k,n}(x) \right)$$

$$= S_1 + S_2$$

Now we find an estimate for S. Using the Cauchy-Schwarz inequality, then considering Lemma C we obtain:

$$S_1 = \frac{1}{b_n}\left(\frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} (k-b_n x)^2 P_{k,n}(x) \right)^{1/2}$$

$$= \frac{1}{b_n}\left(\frac{a_n^2 b_n^2 x^4 + b_n x}{1+a_n x)^2} \right)^{1/2}$$

$$\le c_2 n^{-1/3}.$$

The estimation of S_2 is an easy consequence of Lemma C, if δ is chosen small enough:

$$S_2 = \frac{1}{(1+a_n x)^n} \sum_{\frac{k}{b_n}\ge 2A} \left|\frac{k}{b_n}-x\right| P_{k,n}(x)$$

$$\le \frac{1}{(1+a_n x)^n} \sum_{\left|\frac{k}{b_n}-x\right|\ge\delta} \left|\frac{k}{b_n}-x\right| P_{k,n}(x)$$

$$\le \frac{1}{(1+a_n x)^n b_n^2} \sum_{k=0}^{n} (k-b_n x)^2 P_{k,n}(x)$$

$$\le \frac{1}{b_n^2} \frac{a_n^2 b_n^2 x^4 + b_n x}{(1+a_n x)^2}$$

$$\le n^{-2/3} x^4 + n^{-2/3} x$$

$$\le c_3 n^{-2/3}.$$

Substituting S_1 and S_2 in S, by (4) we obtain the desired result.

E.V.Voronowskaja [5] proved for the Bernstein polynomials that

$$B_n(f,x) = f(x) + \frac{f''(x)}{2n} x(1-x) + \frac{\rho n}{n}, \tag{5}$$

if f(x) is bounded in [0,1], and has a finite second derivative at a certain point x of [0,1]. In (5) ρ_n tends to zero with $n \to \infty$..

Now we prove an asymptotic approximation theorem similar to (5) for Berstein-Balazs type rational functions defined in (2).

Theorem 2. Let f(t) be a function defined in $[0,\infty)$, for which $f(t)=O(e^{\alpha t})(t\to\infty,\alpha$ fixed), then at each point t=x, for which $f''(t)$ exists and is finite

$$R_n^*(f,x)-f(x)=f''(x)\frac{a_n^2b_n^2x^4+x}{2b_n(1+a_nx)^2}+r_n,$$

where $r_n\to 0$ $a_n=\frac{b_n}{n}\to 0$ and $\frac{n^{1/2}}{b_n}\to 0$, as $n\to\infty$.

Proof. Since f'' exists we can write

$$f\left(\frac{k}{b_n}\right)=f(x)+f'(x)\left(\frac{k}{b_n}-x\right)+\left[\frac{f''(x)}{2}+\lambda\left(\frac{k}{b_n}\right)\right]\left(\frac{k}{b_n}-x\right)^2$$

where $\lambda\left(\frac{k}{b_n}\right)\to 0$ if $\frac{k}{b_n}\to x$.

Substituting this expression in $R_n^*(f,x)$ and taking into account the identities in Lemma C we get

$$R_n^*(f,x)=\frac{f(x)}{(1+a_nx)^n}\sum_{k=0}^{n}P_{k,n}(x)+\frac{f'(x)}{(1+a_nx)^nb_n}\sum_{k=0}^{n}(k-b_nx)P_{k,n}(x)$$

$$+\frac{f''(x)}{2(1+a_nx)^nb_n^2}\sum_{k=0}^{n}(k-b_nx)^2P_{k,n}(x)$$

$$+\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\lambda\left(\frac{k}{b_n}\right)\left(\frac{k}{b_n}-x\right)^2P_{k,n}(x)+\frac{a_nx^2}{1+a_nx}f'(x)$$

$$=f(x)+\frac{f''(x)a_n^2x^4+\frac{x}{b_n}}{2\ (1+a_nx)^2}+r_n$$

$$r_n=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n}\lambda\left(\frac{k}{b_n}\right)\left(\frac{k}{b_n}-x\right)^2P_{k,n}(x).$$

It can be easily seen that $r_n\to 0$ as $n\to\infty$ (see [1]), so the theorem is proved.

Now we prove that the derivatives of the functions in (2) converge to the derivative of function. In [1], C. Balázs proved that the derivatives of Bernstein type rational functions, $R_n'(f;x)$ converge to derivative of the function, $f'(x)$, when the interval of convergence is $[0,A]$.

Theorem 3. Let $f \in C[0,\infty)$ and let the derivative f'' exists such that the inequality $|f''(t)| \le M_{f''} e^{\alpha t}$ holds. Then, for $a_n = b_n / n$ and $b_n = n^{2/3}$,

$$\lim_{n\to\infty}\left|\frac{\partial}{\partial x}R_n^*(f;x) - f'(x)\right| = 0.$$

Proof. First consider the case $x > 0$: Obviously, we get

$$\frac{\partial}{\partial x}R_n^*(f;x) = R_n'(f;x) + \frac{2a_n x + a_n^2 x^2}{(1+a_n x)^2} f'(x) + \frac{a_n x^2}{1+a_n x} f''(x) \tag{6}$$

where $R_n'(f;x)$ is derivatives of Bernstein type rational functions, by defined (1). Using Lemma C, it can be seen that

$$R_n'(f;n) = \frac{1}{x(1+a_n x)^n} \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right) P_{k,n}(x)(k - b_n x)$$

$$+ \frac{a_n b_n x}{(1+a_n x)^{n+1}} \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right) P_{k,n}(x).$$

It is known that (see [1])

$$R_n'(f,x) \to f'(x); n \to \infty$$

when $x > 0$. Thus, from (6) we can write

$$\left|\frac{\partial}{\partial x}R_n^*(f;x) - f'(x)\right| \le \frac{2a_n x + a_n^2 x^2}{(1+a_n x)^2}|f'(x)| + \frac{a_n x^2}{1+a_n x}|f''(x)| + |R_n'(f,x) - f'(x)|.$$

Since the derivatives are bounded and from (7) we obtain

$$\left|\frac{\partial}{\partial x}R_n^*(f;x) - f'(x)\right| \to 0 \quad \text{for } n \to \infty.$$

If x=0, C. Balazs [1] showed that

$$R_n'(f;x)\big|_{x=0} \to f'(0).$$

Therefore

$$\lim_{n\to\infty}\left|\frac{\partial}{\partial x}R_n^*(f;x) - f'(x)\right| = 0$$

which completes the proof

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