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r^m – TYPE SOLUTIONS FOR A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Some fundamental solutions of r^m - type for a class of iterated elliptic equations are given, including Laplace equation and its iterates.

1. INTRODUCTION

Much of physical problems are solved in spherical or cylindrical domains. That means, most of the time, the solutions are symmetric functions with respect to a point or with respect to an axis. That is why, when investigating the solutions, we see a frequent use of the type of solutions in terms of a variable r defining a distance to a point. For some of the research have been done for various type of problems, we refer to the references [1 – 5] Here in this study, we apply the idea to a class of linear partial differential equations of second order and its iterates. The class of equations under consideration is

$$Lu = \sum_{i=1}^{n} \left(\frac{r}{x_i} \right)^p \left[x_i^2 \frac{\partial^2 u}{\partial x_i^2} + \alpha_i x_i \frac{\partial u}{\partial x_i} \right] + \lambda u = 0 \tag{1.1}$$

where $\lambda, \alpha_i (i = 1,2,...,n)$ are real parameters, $p(> 0)$ is a real constant and r is defined by

$$r^p = x_1^p + x_2^p + ... + x_n^p. \tag{1.2}$$

The domain of the operator L is the set of all real valued functions $u(x)$ of the class $C^2(D)$, where $x = (x_1, x_2, ..., x_n)$ denotes points in R^n and D is a regularity domain of u in R^n.

When we investigate the solutions of the equation (1.1), in the form of $u = f(r^m), f \in C^2$, we arrive at an Euler equation of second order

$$L[f(r^m)] = m^2 v^2 f''(v) + m(m - p + n(p-1) + \sum_{i=1}^{n} \alpha_i) v f'(v) + \lambda f(v) = 0,$$

with $v = r^m$. Since, the complete solutions of this Euler equation can be derived easily, we conclude from here that, the functions in the form

$$f(r^m) = r^{mc}$$

(and the linear combinations) give solutions to (1.1), where c is a root of the characteristic equation $m^2c^2 + m(-p + n(p-1) + \sum_{i=1}^{n} \alpha_i)c + \lambda = 0$. This result suggests us to investigate the solutions of r^m-type, which will be the subject of next section.

2. r^m-Type Solutions

We first give some properties of the operator L. By a direct computation, is can be shown that

$$L(r^m) = (m(m+\phi) + \lambda) r^m \tag{2.1}$$

where m is a real or complex parameter and

$$\phi = -p + n(p-1) + \sum_{i=1}^{n} \alpha_i. \tag{2.2}$$

The proof of the following lemma can be done easily by using induction argument on k.

Lemma 1. Let L be given by (1.1). If a function u has continuous derivatives of any order with respect to the variables $x_1, x_2, \ldots, x_n$, and with respect to the parameter m, then

$$\frac{\partial^i}{\partial m^i}\left(L^k u\right) = L^k\left(\frac{\partial^i}{\partial m^i} u\right) \tag{2.3}$$

where i is any positive integer and L^k denotes, as usual, the successive applications of the operator L onto itself, that is $L^k u = L(L^{k-1}u)$, where k is a positive integer.

Now, we investigate the solutions of r^m-type of the class of the equations $L^k u = 0$. Let us denote the coefficient of r^m in the equation (2.1) by $\beta(m)$, that is, let

$$\begin{aligned} \beta(m) &= m(m+\phi) + \lambda \\ &= m^2 + \phi m + \lambda \end{aligned} \tag{2.4}$$

Hence for $\Delta = \phi^2 - 4\lambda$, we have three cases:

1. For $\Delta > 0$, there are two different real roots m_1 and m_2 of the polynomial $\beta(m)$,
2. For $\Delta = 0$, there is a multiple root $m_1 = m_2 = w$ of the polynomial $\beta(m)$, and
3. For $\Delta < 0$, there are two conjugate complex roots $m_1 = w_1 + iw_2$ and $m_2 = w_1 - iw_2$ of the polynomial $\beta(m)$, where

$$m_1 = \left(-\phi + \sqrt{\Delta}\right)/2,\ m_2 = \left(-\phi - \sqrt{\Delta}\right)/2, \tag{2.5}$$

$$w = -\phi/2, w_1 = -\phi/2 \text{ and } w_2 = \frac{1}{2}\sqrt{-\Delta}$$

Now, we are ready to give the following theorem.

Theorem 1. A fundamental solution for the iterated equation $L^k u = 0$ can be given by

i. if $\Delta > 0$,

$$u = \sum_{j=0}^{k-1}\left[A_j r^{m_1} + B_j r^{m_2}\right](\ln r)^j,$$

ii. if $\Delta = 0$,

$$u = r^w \sum_{j=0}^{2k-1} C_j (\ln r)^j,$$

or

iii. if $\Delta < 0$,

$$u = r^{w_1}\sum_{j=0}^{k-1}(\ln r)^j\left[D_j \cos(w_2 \ln r) + E_j \sin(w_2 \ln r)\right],$$

where $A_j, B_j, C_j, D_j, E_j, F_j$ are arbitrary constants.

Proof. By applying the operator L successively to the equation (2.1), clearly one has

$$L^k(r^m) = \beta^k(m) r^m. \tag{2.6}$$

Now,

i. Let $\Delta > 0$. Hence by (2.6),

$$L^k(r^m) = (m - m_1)^k (m - m_2)^k r^m. \tag{2.7}$$

Thus, for $m = m_1$ and for $m = m_2$ we have $L^k(r^{m_1}) = 0$ and $L^k(r^{m_2}) = 0$, which means that the functions r^{m_1} and r^{m_2} are solutions of the equation $L^k u = 0$. Under the derivation of the expression (2.6) with respect to the parameter m, the left hand side gives

$$\frac{\partial}{\partial m}\left(L^k(r^m)\right) = L^k\left(\frac{\partial}{\partial m}(r^m)\right) = L^k(r^m \ln r)$$

and the right hand side yields

$$\frac{\partial}{\partial m}\left(\beta^k(m)r^m\right)=k\beta^{k-1}(m)\beta'(m)r^m+\beta^k(m)r^m\ln r$$
$$=\beta^{k-1}(m)\left(k\beta'(m)r^m+\beta(m)r^m\ln r\right)$$

Hence, setting $\theta_1(m)=k\beta'(m)r^m+\beta(m)r^m\ln r$, we obtain

$$L^k(r^m\ln r)=\beta^{k-1}(m)\theta_1(m)=(m-m_1)^{k-1}(m-m_2)^{k-1}\theta_1(m) \qquad (2.8)$$

and from this we conclude that the functions $r^{m_1}\ln r$ and $r^{m_2}\ln r$ are also solutions of the equation $L^k u=0$.

Once again, deriving (2.8) with respect to m yields

$$L^k\left(r^m(\ln r)^2\right)=\beta^{k-2}(m)\theta_2(m)=(m-m_1)^{k-2}(m-m_2)^{k-2}\theta_2(m)$$

where $\theta_2(m)=(k-1)\beta'(m)\theta_1(m)+\beta(m)\theta_1'(m)$, which, in turns, gives that the functions $r^{m_1}(\ln r)^2$ and $r^{m_2}(\ln r)^2$ are solutions of the equation $L^k u=0$.

Proceeding in this way, by taking the derivative with respect to m, (k-2) times in (2.8), finally, we get

$$L^k\left(r^m(\ln r)^{k-1}\right)=m\beta(m)\theta_{k-1}(m)$$

where $\theta_{k-1}(m)=2\beta'(m)\theta_{k-2}(m)+\beta(m)\theta'_{k-2}(m)$. Hence, $r^{m_1}(\ln r)^{k-1}$ and $r^{m_2}(\ln r)^{k-1}$ are solutions of the equation $L^k u=0$. Thus, by the principle of superposition, a complete solution of the equation $L^k u=0$ can be given by

$$u=\sum_{j=0}^{k-1}(A_j r^{m_1}+B_j r^{m_2})(\ln r)^j.$$

ii. Let $\Delta=0$. Hence by (2.6),

$$L^k(r^m)=(m-w)^{2k}r^m$$

where $w=m_1=m_2$ is the multiple root of $\beta(m)$. It is obvious from (2.9) that r^w is a solution of $L^k u=0$. In addition, by taking the derivative with respect to m successively, and using the idea of the preceding proof we conclude that the functions $r^w(\ln r)^j$, $(j=1, 2, \ldots, 2k-1)$ are also solutions of $L^k u=0$. Hence by the principle of superposition the function

$$u=r^w\sum_{j=0}^{2k-1}C_j(\ln r)^j$$

gives a complete solution.

iii. Let $\Delta<0$. By the case (i), we know that the functions $r^{m_1}(\ln r)^j$, and $r^{m_2}(\ln r)^j$, $(j=0, 1, \ldots, k-1)$ are complex valued solutions of $L^k u=0$. Now, to select out the real valued solutions from those, are remember the Euler formula

$$r^m=r^{w_1}r^{\pm iw_2}$$

$$= r^{w_1} e^{\pm i w_2 \ln r}$$
$$= r^{w_1}\left[\cos(w_2 \ln r) \pm i \sin(w_2 \ln r)\right]$$

From here, we can select real valued solutions for $L^k u = 0$ as

$$\left[r^{w_1} \cos(w_2 \ln r)\right](\ln r)^j, \left[r^{w_1} \sin(w_2 \ln r)\right](\ln r)^j, \ (j = 0, 1, \ldots, k-1).$$

Thus, the linear combination

$$u = r^{w_1} \sum_{j=0}^{k-1} (\ln r)^j (D_j \cos(w_2 \ln r) + E_j \sin(w_2 \ln r))$$

gives a complete solution of $L^k u = 0$.

In one dimensional case, the equation (1.1) becomes an Euler (equidimensional) equation of second order and hence r^m type solutions are replaced by x^m type solutions, as were expected. For if $n = 1$, then $r = x_1$ and letting $x_1 = x$, $\alpha_1 = \alpha$, the equation (1.1) is replaced by the Euler equation

$$Eu = x^2 \frac{d^2 u}{dx^2} + \alpha x \frac{du}{dx} + \lambda u = 0$$

and hence for $\beta(m) = m^2 + (\alpha - 1)m + \lambda$, we have $\Delta = (\alpha - 1)^2 - 4\lambda$,

$$m_1 = \left(1 - \alpha + \sqrt{\Delta}\right)/2, \ m_2 = \left(1 - \alpha + \sqrt{\Delta}\right)/2, \ w = (1 - \alpha)/2 = w_1 \text{ and } w_2 = \frac{1}{2}\sqrt{-\Delta}.$$

Thus we can give the following result for the solutions of the iterated Euler equation.

Theorem 2. The general solution of the iterated equation $E^k u = 0$ can be given by

i. if $\Delta > 0$,

$$u = \sum_{j=0}^{k-1} (A_j x^{m_1} + B_j x^{m_2})(\ln x)^j,$$

ii. if $\Delta = 0$,

$$u = x^w \sum_{j=0}^{2k-1} C_j (\ln x)^j,$$

or

iii. if $\Delta < 0$,

$$u = x^{w_1} \sum_{j=0}^{k-1} (\ln x)^j (D_j \cos(w_2 \ln x) + E_j \sin(w_2 \ln x)),$$

where A_j, B_j, C_j, D_j, E_j, F_j are arbitrary constants.

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